CHARACTERIZATIONS OF P(Q(4,q),L)

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In [2] we dealt with a characterization of the generalized quadrangle Q(4,q), q odd, by introducing the concept of (0,2)-set. The aim of this paper is to give a characterization of P(Q(4,q),L), q odd and L an arbitrary regular line of Q(4,q), by constructing these (0,2)-sets and using the result of [2].

I. INTRODUCTION

1.1. Definitions

A finite generalized quadrangle is an incidence structure $S = (P,B,I)$ in which $P$ and $B$ are disjoint (nonempty) sets of objects called points and lines, and for which $I$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y,M) \in P \times B$ for which $x \not I y \not I L$.

Generalized quadrangles were introduced by J. Tits. The integers $s$ and $t$ are the parameters of the generalized quadrangle and $S$ is said to have order $(s,t)$; if $s = t$, $S$ is said to have order $s$. There is a point-line duality for generalized quadrangles (of order $(s,t)$) for which in any definition or theorem the words "point" and "line" and the parameters $s$ and $t$. 
are interchanged.

A grid is an incidence structure $S' = (P', B', I')$ with $P' = \{x_{ij} \mid i = 0, \ldots, s_1 \text{ and } j = 0, \ldots, s_2\}$, $s_1 > 0$ and $s_2 > 0$, $B' = \{L_0, \ldots, L_{s_1}, M_0, \ldots, M_{s_2}\}$, $x_{ij} \in I' L_k$ iff $i = k$, and $x_{ij} \in I' M_k$ iff $j = k$. A grid with parameters $s_1, s_2$ is a generalized quadrangle iff $s_1 = s_2$. Evidently the grids with $s_1 = s_2$ are the generalized quadrangles with $t = 1$.

Let $S = (P, B, I)$ be a generalized quadrangle. For $x, y \in P$, we write $x \sim y$ and say that $x$ and $y$ are collinear, provided that there is some line $L \in B$ for which $x \in L \cap y$. And $x \not\sim y$ means that $x$ and $y$ are not collinear. Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as $L$ and $M$ are concurrent or non-concurrent, respectively.

The line (resp. point) which is incident with distinct collinear points $x, y$ (resp. distinct concurrent lines $L, M$) is denoted by $xy$ (resp. $LM$ or $L \cap M$).

For $x \in P$ put $x^1 = \{y \in P \mid y \sim x\}$, and note that $x \in x^1$. The trace of a pair of distinct points $(x, y)$ is defined to be the set $x^1 \cap y^1$ and is denoted $tr(x, y)$ or $(x, y)^1$. We have $|(x, y)^1| = s+1$ or $t+1$ according as $x \sim y$ or $x \not\sim y$. More generally, if $A \subseteq P$, we define $A^1 = \cap \{x^1 \mid x \in A\}$.

For $x \not\sim y$, the span of the pair $(x, y)$ is $\sigma(x, y) = \{x, y\}^{11} = \{u \in P \mid u \in z^1 \forall z \in x^1 \cap y^1\}$. If $x \not\sim y$, then $(x, y)^{11}$ is also called the hyperbolic line defined by $x$ and $y$. There holds $|(x, y)^{11}| = s+1$ if $x \sim y$ and $|(x, y)^{11}| = t+1$ if $x \not\sim y$.

A pair $(x, y)$ is called regular iff $x \sim y$, with $x \not\sim y$, or if $x \not\sim y$ and $|(x, y)^{11}| = t+1$. The point $x$ is regular provided $(x, y)$ is regular for all $y \in P$, $y \not\sim x$. A point $x$ is coregular provided each line incident with $x$ is regular. The pair $(x, y)$, $x \not\sim y$, is antiregular provided $|z^1 \cap (x, y)^1| < 2$ for all $z \in P \setminus \{x, y\}$.

A point $x$ is antiregular provided $(x, y)$ is antiregular for all $y \in P \setminus x^1$. A triad of points is a triple of pairwise non-collinear points. Given a triad $T = (x, y, z)$, a center of $T$ is just a point of $T^1$. We say $T$ is acentric or centric according as $|T^1|$ is zero or positive. An ovoid of $S$ is a set of $1+st$ points of $S$ that are pairwise non-collinear, i.e. the points of an ovoid partition the lines of $S$. 