RECENT RESULTS ON \((m,n)\)-TYPE \(k\)-SETS IN AN AFFINE PLANE \(\alpha_q\)

Dedicated to Professor R. ARTZY on the occasion of his 75th birthday

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In an affine plane \(\alpha\) of order \(q\), where \(q\) is not necessarily a prime power, let \(K\) be a \((m,n)\)-type \(k\)-set, with \(1 \leq m < n \leq q-1\), that is a set of \(k\) points such that every line meets it either in \(m\) or in \(n\) points.

Set \(a = n - m\). It is known [9] that:

1. \(n + mq \leq k \leq (n-1)q + m\),
2. \(a = n - m, q = ab, m = sa, n = (s+1)a, k = ac\),

with \(a, b, c, s \in \mathbb{N} - \{0\}, 2 \leq a \leq q-2, 1 \leq s \leq m/2, 2 \leq b \leq q/2\),

where \(c\) satisfies the equation:

3. \(c^2 - c[(2s+1)(q+1)-b] + s(s+1)q(q+1) = 0\).

We say that a \((m,n)\)-type \(k\)-set is arithmetically admissible in \(\alpha\) if the quadruple \((k,m,n,q)\) satisfies the conditions (1), (2), (3). In this paper we determine the values of \((k,m,n,q)\) in order that such sets exist, that is we completely characterize \((m,n)\)-type \(k\)-sets in an affine plane from the arithmetic point of view.

Set

4. \(x = c - s(q+1)\).

By (3) we obtain:

5. \(x^2 - x(q+1-b) + s(q+1)(b-s-1) = 0\).

The complement of \(K\) is a \((m',n')\)-type \(k'\)-set \(K'\), with:
(6) \[ k' = q^2 - k = a(qb-c), \quad m' = q-n = a(b-s-1), \]
\[ n' = q-m = a(b-s). \]

Set

(7) \[ s' = b-s-1, \quad c' = qb-c, \quad x' = c'-s'(q+1). \]

With regard to \( K' \) we have:

\[
\begin{cases}
  a' = n'-m' = a, \\
  q = ab, \\
  m' = s'a, \\
  n' = (s'+1)a, \\
  k' = ac', \\
  1 \leq s' \leq m'/2,
\end{cases}
\]

moreover it is:

\[ x'^2 - x'(q+1-b) + s(q+1)s' = 0, \]

so that \( x \) and \( x' \) are the two roots of the equation:

(9) \[ X^2 - X(q+1-b) + s(q+1)s' = 0, \]

hence \( x + x' = q+1-b. \)

By (4) and (2) we obtain:

(10) \[ k = ax + m(q+1). \]

By (7), we have, since \( s \geq 1, \quad s' \geq 1 \) (see (2), (8)):

(11) \[ s + s' = b - 1 \geq 2 \]

and then

(12) \[ b \geq 3, \quad s \leq b-2, \quad s' \leq b-2. \]

Equation (9) must have positive integer roots. Then \( A = (q+1-b)^2 - 4 ss'(q+1) \) is the square of a positive integer \( A \), that is:

(13) \[ A^2 = (q+1-b)^2 - 4 ss'(q+1). \]

By (13), since also \( s' = b-s-1 \), we easily obtain:

(14) \[ A^2 = B^2 - 4s(s+1)s'(s'+1) = B^2 - 4\sigma, \]

where we set

(15) \[ B = q-b+1-2ss', \quad \sigma = s(s+1)s'(s'+1). \]

By (14) we have:

\[ B^2 - A^2 = 4\sigma \iff (B-A)(B+A) = 4\sigma, \]

it follows that:

(16) \[
\begin{cases}
  B - A = D, \\
  B + A = 4\sigma/D,
\end{cases}
\]