A NEW APPROACH TO THE LIBRATIONAL SOLUTION IN THE IDEAL RESONANCE PROBLEM

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Abstract. The librational motion of the Ideal Resonance Problem (Garfinkel, 1966, Jupp, 1969) is treated through an initial non-canonical transformation which, however, leaves the equations of motion in a 'quasi-canonical' form, with Hamiltonian expressed in standard trigonometric functions amenable to traditional averaging techniques. The perturbed solutions, similarly expressed in trigonometric near-identity transformations, and their frequencies can be found to arbitrary order, with the elliptic integrals expected of the system introduced only in a final explicit quadrature for a Kepler-type equation in the angular variable. The specific transformations and resulting equations of motion are introduced, and explicit solutions for the original variables are found to second order, with 'mean motion' accurate to fifth. Limitation of the present solution to the librational region, the extension of that solution to higher order, and observations on the form of the associated Hamiltonian are also discussed.

1. Introduction

Resonance, a pathological situation in mechanical systems, occurs physically with disturbing frequency in our own solar system. For example, the periods of rotation and revolution of Mercury about the Sun, and the Moon about the Earth, are small whole-number ratios of one another. And the periods of revolution of Neptune and Pluto, of some of the Jovian satellites, and of Jupiter itself and Saturn, as well as certain minor planets, are also nearly commensurate. It also manifests itself analytically in artificial satellite theory, including problems involving the critical inclination. When the Hamiltonians of such systems are analyzed through a perturbation analysis, linear combinations of their frequencies (which inevitably find their way into the denominators of the canonical transformations) become sufficiently small to threaten even the formal, asymptotic 'convergence' of the resulting solutions.

In a seminal paper, Garfinkel (1966) indicated how, in perturbed system with a single resonance, the 'critical argument' generating such linear combinations could be isolated by using traditional methods to eliminate the less pathological terms, then reducing the system, through a final canonical transformation, to one of the form

\[-F = B(y) + 2\mu^2 A(y) \sin^2 x\]

(1.1)

in which \(\mu\) is the perturbation parameter, the coordinate \(x\) is the critical argument, and \(y\) its conjugate momentum. (The negative sign on the Hamiltonian is retained for reasons of historical convention.) This system, which he characterized as the ideal resonance problem, was later generalized (Garfinkel, 1976) to ones of the form

\[-F = B(y) + 2\mu^2 A(y)f(x)\]

(1.1')

in which \(f(x)\) is of period \(2\pi\).

The single-degree-of-freedom system (1.1) has been the starting point for a number of papers by Garfinkel himself, as well as by Jupp and Williams. It is customary to restrict
attention to the \textit{normal} resonance problem (Garfinkel, 1966, 1972b);

$$0 < B'' = O(B), \quad A = O(B).$$

In the resonance region, this specializes to:

$$A = O(1), \quad B'' = O(1). \quad (1.2)$$

(Garfinkel, 1972b, paraphrased in 1972a, equation (6)).

The initial step of the analysis is a Taylor expansion of the functions $A$ and $B$ about some point:

$$A(y) = A(y_0 + p) = A^{(0)} + A^{(1)} p + \frac{1}{2} A^{(2)} p^2 + \frac{1}{6} A^{(3)} p^3 + \cdots$$

$$B(y) = B(y_0 + p) = B^{(0)} + B^{(1)} p + \frac{1}{2} B^{(2)} p^2 + \frac{1}{6} B^{(3)} p^3 + \cdots \quad (1.3)$$

(Superscripts conventionally denote the order of differentiation, while the zero subscripts indicate evaluation of those derivatives at $y_0$; the latter will be dropped in the sequel). Constant terms are irrelevant in the Hamiltonian system, and if, following Jupp (1969, Section II), $y_0$ is chosen to be a \textit{libration center} such that $B^{(1)} = 0$, the problem reduces to

$$- H(x, p) = \frac{1}{2} B^{(2)} p^2 + \frac{1}{6} B^{(3)} p^3 + \cdots$$

$$+ 2\mu^2 (A^{(0)} + A^{(1)} p + \frac{1}{2} A^{(2)} p^2 + \cdots) \sin^2 x. \quad (1.4)$$

(Note that, as a simple unscaled translation, the transformation $(x, y) \leftrightarrow (x, p)$ is itself canonical.)

A perturbation analysis typical consists of expanding the full, perturbed solution about the ‘unperturbed’ one, which thus must be found first. For $p = O(1)$, this comprises terms surviving when $\mu \to 0$; but in the vicinity of the libration center, $p = 0(\mu)$, the ‘perturbation’ and ‘unperturbed system’ are of the same order and the very character of the solution is altered. Thus the unperturbed problem is identified with the dominant terms

$$\frac{1}{2} B^{(2)} p^2 + 2\mu^2 A^{(0)} \sin^2 x. \quad (1.5)$$

The analogy with the [finite oscillations of the] simple pendulum, with its librational and circulatory regions separated by a separatrix, was early recognized, and this analogy has formed the basis for subsequent analysis of the ideal resonance problem. Jupp, in particular, starts from the libration region, expanding the Hamiltonian about its equilibrium point (1969) and ultimately introduces Lie transforms to obtain explicit solutions through second order in the librational region and through first order beyond (1972, 1974). This solution thus becomes implicitly limited to regions in the vicinity of the libration center, though it does extend outside the strictly librational region. Garfinkel, on the other hand, expands $A$ and $B$ in (1.3) about an \textit{arbitrary} point, $y_0'$, say, in the transformed $(x', y')$ phase plane. A classical mixed-variable canonical transformation is then used to eliminate $x'$ from the transformed Hamiltonian, making $y'$