\textbf{HSP \neq SHPS for metabelian representable l-groups}

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\textit{Abstract.} For the standard operators on classes of algebras, \( H \) (homomorphic images), \( S \) (subalgebras) and \( P \) (products) it is shown by counterexamples that \( \text{HSP} \neq \text{SHPS} \) and \( HP \not\preceq \text{SPHS} \) for metabelian representable \( l \)-groups. From these two inequalities it follows that the partially ordered semigroup of operators on metabelian representable \( l \)-groups generated by \( \{H, P, S\} \) is the "standard" 18-element diagram.

Let \( X \) be a class of lattice-ordered groups (\( l \)-groups) such that if \( H \cong G \in X \) then \( H \in X \). We shall write \( H(X) \) for the class of all homomorphic images of \( l \)-groups in \( X \), \( S(X) \) for the class of \( l \)-subgroups of \( l \)-groups in \( X \), and \( P(X) \) for the class of direct products of families of \( l \)-groups in \( X \). In this paper it is shown that \( \text{HSP} \neq \text{SHPS} \) for metabelian representable \( l \)-groups (Theorem 1). This result gives a negative solution of question 28(ii) of [1]. Also it is shown (Theorem 2) that \( HP \not\preceq \text{SPHS} \) for metabelian representable \( l \)-groups and so with the use of results of [5] it follows that the partially ordered semigroup of operators on metabelian representable \( l \)-groups generated by \( \{H, P, S\} \) is the "standard" 18-element algebra.

In this paper \( N \) denotes the set of natural numbers. As usual, \( x \gg y \) means that \( x > y^n \) for every \( n \in N \). All basic facts and definitions about lattice-ordered groups and universal algebra can be found in [2], [3] respectively.

\textbf{1. Preliminaries}

Let \( R \) be the linearly ordered additive group of real numbers and \( R^+ \) be the multiplicative group of positive real numbers. Let \( T_\beta \) be the set

\[ T_\beta = \{(r, a) \mid r \in (\beta) \subseteq R^+, a \in A_\beta \subseteq R\} \]

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where \((\beta)\) is the infinite cyclic subgroup of \(R^+\) generated by the positive number \(\beta\), and \(A_\beta\) is any subgroup of \(R\) containing 1 and closed under multiplication by \(\beta\) and \(\beta^{-1}\), with operation \(\cdot\) defined by the rule

\[(r, a) \cdot (r', a') = (rr', r'a + a').\]

It is clear that the set \(T_\beta\) with the operation \(\cdot\) is a group. Now define a linear order on the group \(T_\beta\). Let \(T_\beta \ni (r, a) > e\) iff \(r = \beta p\) and \(p > 0\) or \(r = 1\) and \(a > 0\) in \(R\).

By \(N_k\) we denote the nilpotent group of class \(k\) defined by the following relations:

\[N_k = \text{grp} \langle b, a_1, a_2, \ldots, a_k \mid [a_1, b] = a_2, [a_2, b] = a_3, \ldots, [a_{k-1}, b] = a_k, [a_i, a_j] = e \ (1 \leq i < j \leq k) \rangle\]

with linear order \(N_k \ni b\sigma a_1^{i_1} \cdots a_k^{i_k} > e\) iff \(m > 0\) or \(m = 0\) and \(t_1 = t_2 = \ldots = t_{i-1} = 0\) and \(t_i > 0\) for some \(1 \leq i \leq k\).

Now let us consider the positive real numbers \(\beta_n = n/n + 1\) \((n \in \mathbb{N})\) and \(G = \prod_{\pi \in N} T_\beta_n\), the full direct product of the \(o\)-groups \(T_\beta_n\). It is evident that:

1. The subgroup \(K = \prod_{\pi \in N} A_\beta_n\), where \(A_\beta_n = \{(1, a) \mid a \in A_\beta_n\}\) is an \(l\)-ideal in \(G\).
2. Every element of \(K\) can be represented as an infinite sequence

\[k = ((1, a_1); (1, a_2); \ldots; (1, a_n); \ldots).\]

Let \(G_0\) be the \(l\)-subgroup of \(G\) generated by the element

\[g = ((\beta_1, 0); (\beta_2, 0); \ldots; (\beta_n, 0); \ldots)\]

and the \(l\)-ideal \(K\). Now consider the set

\[H = \{k = ((1, a_1); (1, a_2); \ldots; (1, a_n); \ldots) \in K \mid \lim_{n \to \infty} a_n = 0\}.\]

**Lemma 1.** \(H\) is an \(l\)-ideal in the \(l\)-group \(G_0\).

The proof is straightforward.

**Lemma 2.** The factor \(l\)-group \(G_0/H\) contains a copy of the linearly ordered group \(N_k\) for every \(k \in \mathbb{N}\).