On Fixed Points of Conformal Pseudogroups
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Abstract. We show in this paper Theorem 2 that if \((H, H_1)\) is a pseudogroup generated by a finite number \(H_1\) of germs of conformal diffeomorphisms of \(\mathbb{C}\) defined on a sufficiently small disc \(D\), which is not linearizable and such that the linear group \((L, H_1) = \{g'(0)/g \in (H, H_1)\} \subset \mathbb{C}^*\) is dense in \(\mathbb{C}^*\), then the set of fixed points of the pseudogroup \((H, H_1)\) is dense in \(D\). This implies the abundance of distinct homotopy classes of loops in leaves of foliations defined in \(\mathbb{C}^2\) by generic polynomial vector fields as well as for germs of holomorphic vector fields in \(\mathbb{C}^2\) beginning with generic jets, both of degree at least 2. These homotopy classes may be realized arbitrarily close to the line at infinity or to 0, respectively. This shows the genericity of polynomial vector fields with infinite Petrovsky-Landis genus ([5]).

The idea of the proof is very simple. If \(g\) is a non-linear conformal diffeomorphism with multiplier \(\lambda = g'(0)\), then the map obtained by the composition of \(g\) and the linear map with multiplier \(\lambda^{-1}\) will have at 0 a fixed point of multiplicity at least 2. Since we may approximate \(\lambda^{-1}\) by elements \(h\) in the pseudogroup and the multiplicity of fixed points satisfy a law of conservation of number, we obtain that \(h \circ g\) has fixed points close to 0. These fixed points appear as a by product of the relative non-linearity of the generators of the pseudogroup, since linearizable pseudogroups have 0 as an isolated fixed point. The fixed points obtained are not conjugate since they have distinct multipliers.

The main technical tool is the angular derivative introduced in [8]. It allows one to split the search for fixed points into two parts: One is to obtain a contraction and the other is to return arbitrarily close to the starting point without modifying the property of contraction. This is carried out since the angular derivative is multiplicative for compositions and is identically 1 for linear maps.

1. The angular derivative

The angular derivative \(\Delta\) of a conformal diffeomorphism \(g: \text{Dom}(g) \to \mathbb{C}\), \(g(0) = 0\) is defined for \(z \in \text{Dom}(g)\) as:

\[
\Delta g(z) = \frac{z g'(z)}{g(z)}
\]

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The real part of this derivative is \( \frac{\partial}{\partial \theta} \text{Arg}(g(z)) \), where \( z = re^{i\theta} \). An interesting property of this derivative is its compatibility with composition: if \( f \) and \( g \) are conformal diffeomorphisms with \( f(0) = g(0) = 0 \), and if \( z \) is in the domain of \( g \circ f \) then

\[
\Delta(g \circ f)(z) = (\Delta g)(f(z)) \cdot (\Delta f)(z)
\]

We have \( \Delta g(z) = 1 + O(z) \); and \( g \) is linear if and only if \( \Delta g \equiv 1 \).

If \( g_\lambda \) denotes the map obtained by composing \( g \) and the linear map with multiplier \( \lambda \) we have:

\[
g_\lambda(z) = \lambda \cdot g(z), \quad \Delta g_\lambda(z) = \Delta g(z)
\]

The multiplicity of a fixed point \( z_0 \) of \( g \) is the multiplicity as a root of \( g(z) - z \) at \( z_0 \), i.e. it is simple or of multiplicity 1 if \( g(z_0) = z_0 \) and \( g'(z_0) \neq 1 \), and of multiplicity \( m > 1 \) if in local coordinates it can be written as

\[
g(z) = z_0 + (z - z_0) + b_m(z - z_0)^m + O((z - z_0)^{m+1}), \quad b_m \neq 0.
\]

The multiplicity of fixed points satisfies a law of conservation of number, in the sense that under a small perturbation the local sum is preserved. This follows since we may compute this index as the intersection index of the graph of \( g \) with the diagonal.

We use the notation of discs in the complex plane:

\[
D(z_0, r) := \{ z \in \mathbb{C} / |z - z_0| < r \} \quad D_\alpha := D(0, \alpha)
\]

**Lemma 1.** Let \( g : D_\alpha \rightarrow g(D_\alpha) \subset \mathbb{C} \) be a non-linear conformal diffeomorphisms onto its image preserving 0 with power series expansion

\[
g(z) = a_1 z + a_m z^m + \ldots, \quad a_1 \neq 0, m \geq 2, a_m \neq 0
\]

and let \( g_\lambda = \lambda \cdot g \). There exist \( \epsilon > 0 \) such that if \( 0 < |\lambda - a_1^{-1}| < \epsilon \) then \( g_\lambda \) has \( m \) simple fixed points near 0. If \( z_0 \neq 0 \) is a fixed point of \( g_\lambda \), then \( \lambda = \frac{z_0}{g(z_0)} \) and the multiplier of \( g_\lambda \) at \( z_0 \) is \( \Delta g(z_0) \).

**Proof.** Consider the map

\[
G : D_\alpha \times \mathbb{C} \rightarrow \mathbb{C} \quad G(z, \lambda) = \lambda \cdot g(z)
\]