ABOUT THE FIRST INTEGRALS OF THE GENERALIZED PROBLEM OF TRANSLATORY-ROTARY MOTION OF RIGID BODIES

G. N. DUBOSHIN

State Astronomical Institute, Moscow, U.S.S.R.

(Received 29 October, 1971)

Abstract. The existence of ten first integrals for the classical problem of the motion of a system of material points, mutually attracting according to Newtonian law, is well known.

The existence of the analogous ten first integrals for the more complicated problem of the motion of a system of absolutely rigid bodies, whose elementary particles mutually attract according to the Newtonian law, was established by the author (Duboshin, 1958, 1963, 1968).

In his later papers (Duboshin, 1969, 1970), the problem of the motion of a system of material points, attracting each other according to a more general law, was considered and, in particular, it was shown under what conditions the ten first integrals, analogous to the classical integrals, may exist for this problem.

In the present paper, the generalized problem of translatory-rotatory motion of rigid bodies, whose elementary particles acting upon each other according to arbitrary laws of forces along the straight line joining them, is discussed.

The author has shown that the first integrals for this general problem, analogous to the integrals of the problem of the translatory-rotatory motion of rigid bodies, whose elementary particles acting according to the Newtonian law, exist under certain well known conditions.

That is, it has been established that if the third axiom of dynamics (action = reaction) is satisfied, then the integrals of the motion of centre of inertia and the integrals of the moment of momentum exist for this generalized problem.

If the third axiom is not satisfied, then the above mentioned integrals do not exist.

The third axiom is a necessary but not a sufficient condition for the existence of the tenth integral – the energy integral. The tenth integral always exists if the elementary particles of the bodies acting with a force, depend only on the mutual distances between them. In this case the force function exists for the problem and the energy integral can be expressed in a well known form.

The tenth integral may exist for some more general case, without expressing the principle of conservation of energy, but permitting calculation of the kinetic energy, if the configuration of a system is given.

The problem, in which the elementary particles acting according to the generalized Veber's law (Tisserand, 1896) has been cited as an example of this more general case.

1. Statement of the Problem. Equations of Motion

Consider a system of finite number of absolutely rigid bodies \( T_i \) \((i=0, 1, 2, ..., n)\), each having a definite, given structure, invariable shape and constant finite mass \( m_i \). Suppose that each elementary particle of mass \( dm_i \) of the body \( T_i \), concentrated at point \( M_i \) will be assumed under the influence of the force, whose source is the elementary particle \( dm_j \) of the body \( T_j \), located at the point \( M_j \) \((j=0, 1, 2, ..., n; j \neq i)\).

Further it will be supposed that this force is always directed along the straight line passing through points \( M_i \) and \( M_j \) and is proportional to the product of the masses of the particles \( dm_i dm_j \) and a function, \( F_{ij} \), of the time, of the coordinates of these
particles and in general also of their first and second derivatives with respect to time.\(^*\)

If the particle \(M_j\) attracts the particle \(M_i\), then \(F_{ij} > 0\), otherwise \(F_{ij} < 0\).

Functions \(F_{ij}\), defining the law of forces, can be arbitrarily given functions, satisfying only general conditions of existence and uniqueness of the motion of our material system.

Here, in general

\[ F_{ji} \neq F_{ij} \]  

(1)
i.e. the third Newtonian axiom of dynamics is, in general, not satisfied and the force with which the particle \(M_j\) acts on the particle \(M_i\) is not equal to the force with which the particle \(M_i\) acts on the particle \(M_j\).

Take some absolute system of coordinates \((\xi, \eta, \zeta)\) with origin at the arbitrary chosen point \(O\) of the space. The coordinates of the point \(M_i\) of the body \(T_i\) will be denoted by \(\xi_i, \eta_i, \zeta_i\) and let \(\xi_i, \eta_i, \zeta_i\) denote the coordinates of the centre of inertia \(G_i\) of the body \(T_i\). As usual, the mutual distance between the points \(M_i\) and \(M_j\) will be denoted by \(a_{ij}\) and hence we get

\[ a_{ij}^2 = (\xi_j - \xi_i)^2 + (\eta_j - \eta_i)^2 + (\zeta_j - \zeta_i)^2. \]  

(2)

The projections of the force acting on the point \(M_i\) due to the particle at the point \(M_j\) along the axes \((\xi, \eta, \zeta)\), are respectively:

\[ F_{ij} \frac{\xi_j - \xi_i}{a_{ij}} \frac{dM_i}{dm_j}, \quad F_{ij} \frac{\eta_j - \eta_i}{a_{ij}} \frac{dM_i}{dm_j}, \quad F_{ij} \frac{\zeta_j - \zeta_i}{a_{ij}} \frac{dM_i}{dm_j}. \]

Projections of the moment of this force with respect to the centre of inertia \(G_i\) of the body \(T_i\) along the same axes will be:

\[ F_{ij} \frac{(\eta_j - \eta_i)(\zeta_j - \zeta_i) - (\zeta_j - \zeta_i)(\eta_j - \eta_i)}{a_{ij}} \frac{dM_i}{dm_j}, \]

\[ F_{ij} \frac{(\zeta_i - \zeta_i)(\xi_j - \zeta_i) - (\xi_i - \zeta_i)(\zeta_j - \zeta_i)}{a_{ij}} \frac{dM_i}{dm_j}, \]

\[ F_{ij} \frac{(\xi_i - \xi_i)(\eta_j - \eta_i) - (\eta_i - \eta_i)(\xi_j - \xi_i)}{a_{ij}} \frac{dM_i}{dm_j}. \]

Hence the projections of the resultant of all the forces, acting on the body \(T_i\) due to the presence of the body \(T_j\) and applied at the centre of inertia \(G_i\) are determined by the formulas:

\[ Z_{ij} = \int_{(T_i)} \frac{dM_i}{(T_j)} \int F_{ij} \frac{\xi_j - \xi_i}{a_{ij}} \frac{dM_i}{dm_j}, \]

\[ H_{ij} = \int_{(T_i)} \frac{dM_i}{(T_j)} \int F_{ij} \frac{\eta_j - \eta_i}{a_{ij}} \frac{dM_i}{dm_j}, \]

\[ Z_{ij} = \int_{(T_i)} \frac{dM_i}{(T_j)} \int F_{ij} \frac{\zeta_j - \zeta_i}{a_{ij}} \frac{dM_i}{dm_j}. \]  

(3)

* The constant factors of proportionality are supposed to be included in the functions \(F_{ij}\).