ON THE CANONICAL EXTENSION OF A DIFFERENTIABLE MANIFOLD

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In this paper, the differential geometry of second canonical extension $^2M$ of a differentiable manifold $M$ is studied. Some vector fields tangent to $^2M$ in $TTM$ are determined. In addition we obtain that the second canonical extensions of $M$ and a totally geodesic submanifold in $M$ are totally geodesic submanifolds in $TTM$ and $^2M$ respectively.

1 INTRODUCTION AND PRELIMINARIES

The differential geometry of differentiable extensions was first studied by R.H. Bowman [1], [2], [3]. Basic definitions and some results about the extensions are assumed and details can be found in Bowman [1]. Summation over repeated indices is always implied.

Let $M$ be an $m$-dimensional differentiable (≡ $C^\infty$) manifold and $(U, \varphi = (x^i))_{1 \leq i \leq m}$ be a chart of $M$. Then $(\tilde{U}, \tilde{\varphi} = (x^{0i}, x^{1i}))_{1 \leq i \leq m}$ and $(\overline{U}, \overline{\varphi} = (x^{0i}, x^{1i}, x^{2i}, x^{3i}))_{1 \leq i \leq m}$ are induced charts of $TM$ and $TTM$ respectively. The projections $\pi_M : TM \to M$ and $\pi_{TM} : TTM \to TM$ are given respectively by $\pi_M(x^{0i}, x^{1i}) = (x^i)$, $\pi_{TM}(x^{0i}, x^{1i}, x^{2i}, x^{3i}) = (x^{0i}, x^{1i})$. Then the tangential map $d\pi_M$ has local expression $d\pi_M(x^{0i}, x^{1i}, x^{2i}, x^{3i}) = (x^{0i}, x^{2i})$.

The second canonical extension $^2M = \{ A \ | \ A \in TTM, \pi_{TM}(A) = d\pi_M(A) \}$ of $M$ is a 3m-dimensional differentiable manifold in $TTM$. An induced chart on $^2M$ is $(^2U, ^2\varphi = (x^{0i}, x^{1i}, x^{3i}), \pi_{TM}(x^{0i}, x^{1i}, x^{2i}, x^{3i}))$, where for $\alpha = 0, 1, 3$, $x^{a\alpha} = x^{a\alpha} \circ \iota$, the natural inclusion $\iota : ^2M \to TTM$, $\iota(x^{0i}, x^{2i}, x^{3i}) = (x^{0i}, x^{1i}, x^{1i}, x^{3i})$, $^2U = \overline{U} \cap ^2M$, $^2\varphi = Pr_3 \circ \overline{\varphi} \mid _{^2U}$, $Pr_3 : \mathbb{R}^{4m} \to \mathbb{R}^{3m}, Pr_3(u^1, \ldots, u^{4m}) = (u^1, \ldots, u^{2m}, u^{3m+1}, \ldots, u^{4m})$, $\overline{\varphi} \mid _{^2U} = (x^{0i}, x^{1i}, x^{1i}, x^{3i})$. We denote by $d\pi(^2X)(^2M)$ the space of vector fields tangent to submanifold $^2M$ in $TTM$, then we see that $d\pi(^2X)(^2M) = Sp\{ \frac{\partial}{\partial x^{0i}}, \frac{\partial}{\partial x^{1i}}, \frac{\partial}{\partial x^{2i}}, \frac{\partial}{\partial x^{3i}} \}$.

$(^2M, ^2\pi, TM, \mathbb{R}^{4m})$ is a fibre bundle relative to the induced coordinates [4]. Moreover this fibre bundle is a subbundle of $(TTM, \pi_{TM}, TM, \mathbb{R}^{3m})$ and $(TTM, d\pi_M, TM, \mathbb{R}^{2m})$. If $V_A(^2M)$
denotes the vertical space at \( A \in 2 \mathcal{M} \) and \( 2 \mathcal{M}_z \) denotes the fibre at \( z \in TM \), then we have
\[
V_A(2 \mathcal{M}) = S_{\pi} \left\{ \frac{\partial}{\partial x_{3i}} | A | 1 \leq i \leq m \right\},
\]
\[
2 \mathcal{M}_z = \{ A | \tau(A) \in T_z(TM) \} = \{ A | A \in 2 \mathcal{M}, \, 2 \varphi(A) = (x_0^i(p), x_1^i(z), x_3^i(A)) \}
\]
where \( \pi_M(z) = p \) and \( \pi_{TM}(A) = z \).

## 2 Main Results

Lemma 1: The fibre of \((2 \mathcal{M}, 2 \pi, TM, \mathcal{R}^m)\) at a point \( z = a \cdot \frac{\partial}{\partial x^i} | p \) of \( TM \) is expressed as
\[
2 \mathcal{M}_z = a \cdot \frac{\partial}{\partial x^i} | z + V_z(TM)
\]
where \( V_z(TM) \) is the vertical space at \( z \).

Proof: Since \( \tau(A) \) is an element of \( T_z(TM) \) for each \( A \in 2 \mathcal{M}_z \), we have \( r_A = (x_0^i(p), x_1^i(z), x_2^i(A), x_3^i(A)) \) and then, \( A = a \cdot \frac{\partial}{\partial x^i} | z + \xi^i \cdot \frac{\partial}{\partial x^i} | z \), \( \xi^i = x_3^i(A) \).

Since \( \xi^i \cdot \frac{\partial}{\partial x^i} | z \in V_z(TM) \), we get that
\[
2 \mathcal{M}_z = a \cdot \frac{\partial}{\partial x^i} | z + V_z(TM).
\]

From Lemma 1, we conclude

Corollary 1: The fibre of \((2 \mathcal{M}, 2 \pi, TM, \mathcal{R}^m)\) at the point \( z \in TM \) is an intersection of fibres of bundles \((TTM, \pi_{TM}, TM, \mathcal{R}^m)\) and \((TTM, d\pi_{TM}, TM, \mathcal{R}^m)\) at \( z \).

Corollary 2: For each \( z \in TM \), \( 2 \mathcal{M} \cap T_z(TM) \neq \phi \).

By Corollary 2, we have \( \overline{V} \cap 2 \mathcal{M} \neq \phi \) for every open set \( \overline{V} \) in \( TTM \). So if \( \eta \) is a differentiable element on \( TTM \), then we can always talk about restriction of \( \eta \) on \( 2 \mathcal{M} \). For example, if \( \overline{X} = \xi^{ai} \cdot \frac{\partial}{\partial x^{ai}} \in \mathcal{X}(TTM), \, a = 0, 1, 2, 3, \, i = 1, 2, \ldots, m \), then \( \overline{X} \big|_{2 \mathcal{M}} \) can be written as
\[
(\xi^0 \circ \tau) \frac{\partial}{\partial x^{3i}} \circ \tau + \frac{1}{2} (\xi^{3i} \circ \tau + \xi^{2i} \circ \tau) \cdot \left( \frac{\partial}{\partial x^{3i}} + \frac{\partial}{\partial x^{2i}} \right) \circ \tau + (\xi^3 \circ \tau) \cdot \frac{\partial}{\partial x^{3i}} \circ \tau + \frac{1}{2} (\xi^{1i} \circ \tau - \xi^{2i} \circ \tau) \cdot \left( \frac{\partial}{\partial x^{1i}} - \frac{\partial}{\partial x^{2i}} \right) \circ \tau
\]
where \( d\tau(\frac{\partial}{\partial x^{3i}}) = \frac{\partial}{\partial x^{3i}} \circ \tau, \, d\tau(\frac{\partial}{\partial x^{2i}}) = \left( \frac{\partial}{\partial x^{2i}} + \frac{\partial}{\partial x^{3i}} \right) \circ \tau \) and \( d\tau(\frac{\partial}{\partial x^{1i}}) = \frac{\partial}{\partial x^{1i}} \circ \tau \). Hence for the vector field
\[
2X = (\xi^0 \circ \tau) \frac{\partial}{\partial x^{3i}} + \frac{1}{2} (\xi^{1i} \circ \tau + \xi^{2i} \circ \tau) (\frac{\partial}{\partial x^{1i}} + \frac{\partial}{\partial x^{2i}}) + (\xi^3 \circ \tau) \frac{\partial}{\partial x^{3i}}
\]
on \( 2 \mathcal{M} \), we have
\[
\overline{X} \big|_{2 \mathcal{M}} = d\tau(2X) + \frac{1}{2} (\xi^{1i} - \xi^{2i}) \circ \tau \cdot (\frac{\partial}{\partial x^{1i}} - \frac{\partial}{\partial x^{2i}}) \circ \tau
\]