ON THE CANONICAL EXTENSION OF A DIFFERENTIABLE MANIFOLD

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In this paper, the differential geometry of second canonical extension $^2M$ of a differentiable manifold $M$ is studied. Some vector fields tangent to $^2M$ in $TTM$ are determined. In addition we obtain that the second canonical extensions of $M$ and a totally geodesic submanifold in $M$ are totally geodesic submanifolds in $TTM$ and $^2M$ respectively.

1 INTRODUCTION AND PRELIMINARIES

The differential geometry of differentiable extensions was first studied by R.H. Bowman [1], [2], [3]. Basic definitions and some results about the extensions are assumed and details can be found in Bowman [1]. Summation over repeated indices is always implied.

Let $M$ be an $m$-dimensional differentiable ($\equiv C^\infty$) manifold and $(U, \varphi = (x^i))_{1 \leq i \leq m}$ be a chart of $M$. Then $(\overline{U}, \overline{\varphi} = (x^{0i}, x^{1i}))_{1 \leq i \leq m}$ and $(\overline{U}, \overline{\varphi} = (x^{0i}, x^{1i}, x^{2i}, x^{3i}))_{1 \leq i \leq m}$ are induced charts of $TM$ and $TTM$ respectively. The projections $\pi_M : TM \to M$ and $\pi_{TM} : TTM \to TM$ are given respectively by $\pi_M(x^{0i}, x^{1i}) = (x^i)$, $\pi_{TM}(x^{0i}, x^{1i}, x^{2i}, x^{3i}) = (x^{0i}, x^{1i})$. Then the tangential map $d\pi_M$ has local expression $d\pi_M(x^{0i}, x^{1i}, x^{2i}, x^{3i}) = (x^{2i}, x^{3i})$.

The second canonical extension $^2M = \{ A \mid A \in TTM, \pi_{TM}(A) = d\pi_M(A) \}$ of $M$ is a $3m$-dimensional differentiable manifold in $TTM$. An induced chart on $^2M$ is $(^2U, ^2\varphi = (x^{0i}, x^{1i}, x^{3i}))$, where for $\alpha = 0, 1, 3$, $x^{ai} = x^{ai} \circ \iota$, the natural inclusion $\iota : ^2M \to TTM$, $\iota(x^{0i}, x^{1i}, x^{3i}) = (x^{0i}, x^{1i}, x^{1i}, x^{3i})$, $^2U = \overline{U} \cap ^2M$, $^2\varphi = P\pi_3 \circ \overline{\varphi} |_{^2U}$, $P\pi_3 : \mathbb{R}^{4m} \to \mathbb{R}^{3m}$, $P\pi_3(u^1, \ldots, u^{4m}) = (u^1, \ldots, u^{2m}, u^{3m+1}, \ldots, u^{4m})$, $\overline{\varphi} |_{^2U} = (x^{0i}, x^{1i}, x^{1i}, x^{3i})$. We denote by $d\iota(\mathcal{X}(^2M))$ the space of vector fields tangent to submanifold $^2M$ in $TTM$, then we see that $d\iota(\mathcal{X}(^2M)) = SP\{\frac{\partial}{\partial x^{0i}}, \frac{\partial}{\partial x^{1i}} + \frac{\partial}{\partial x^{2i}}, \frac{\partial}{\partial x^{3i}}\}$.

$(^2M, ^2\pi, TM, \mathbb{R}^m)$ is a fibre bundle relative to the induced coordinates [4]. Moreover this fibre bundle is a subbundle of $(TTM, \pi_{TM}, TM, \mathbb{R}^{2m})$ and $(TTM, d\pi_M, TM, \mathbb{R}^{2m})$. If $V_A(^2M)$
denotes the vertical space at \( A \in 2M \) and \( 2M_z \) denotes the fibre at \( z \in TM \), then we have
\[ V_A(2M) = S_r \{ \frac{\partial}{\partial x^{i_1}} | A | \ 1 \leq i \leq m \}, \]
\[ 2M_z = \{ A | \ i(A) \in T_z(TM) \} = \{ A | A \in 2M, \ 2\varphi(A) = (x^{a_0(p)}, x^{a_1(z)}, x^{a_3(A)}) \} \]
where \( \pi_M(z) = p \) and \( \pi_{TM}(A) = z \).

2 MAIN RESULTS

Lemma 1: The fibre of \( (2M, 2\pi, TM, \mathcal{R}^m) \) at a point \( z = a_i \frac{\partial}{\partial x^i} | p \) of TM is expressed as
\[ 2M_z = a_i \frac{\partial}{\partial x^i} \mid_z + V_z(TM) \]
where \( V_z(TM) \) is the vertical space at \( z \).

Proof: Since \( \iota(A) \) is an element of \( T_z(TM) \) for each \( A \in 2M_z \), we have \( \varphi(A) = (x^{a_0(p)}, x^{a_1(z)}, x^{a_2(A)}, x^{a_3(A)}) \) and then, \( A = a_i \frac{\partial}{\partial x^i} \mid_z + \xi^i \frac{\partial}{\partial x^i} \mid_z, \ \xi^i = x^{a_3(A)} \).
Since \( \xi^i \frac{\partial}{\partial x^i} \mid_z \in V_z(TM) \), we get that
\[ 2M_z = a_i \frac{\partial}{\partial x^i} \mid_z + V_z(TM). \]
\[ \square \]

From Lemma 1, we conclude
Corollary 1: The fibre of \( (2M, 2\pi, TM, \mathcal{R}^m) \) at the point \( z \in TM \) is an intersection of fibres of bundles \( (TTM, \pi_{TM}, TM, \mathcal{R}^m) \) and \( (TTM, d\pi_{TM}, TM, \mathcal{R}^m) \) at \( z \).

Corollary 2: For each \( z \in TM \), \( 2M \cap T_z(TM) \neq \emptyset \).

By Corollary 2, we have \( \overline{V} \cap 2M \neq \emptyset \) for every open set \( \overline{V} \) in \( TTM \). So if \( \eta \) is a differentiable element on \( TTM \), then we can always talk about restriction of \( \eta \) on \( 2M \). For example, if \( \overline{X} = \xi^a \frac{\partial}{\partial x^a} \in \mathcal{X}(TTM), \ \alpha = 0, 1, 2, 3, \ \ i = 1, 2, \cdots, m, \) then \( \overline{X} \big|_{2M} \) can be written as
\[ \overline{X} \big|_{2M} = (\xi^{a_i} \circ \iota) \cdot \frac{\partial}{\partial x^{i_a}} + \frac{1}{2} (\xi^{a_i} \circ \iota + \xi^{a_i} \circ \iota) \cdot (\frac{\partial}{\partial x^{a_1}} + \frac{\partial}{\partial x^{a_2}}) \circ \iota + \xi^{a_3} \circ \iota \cdot \frac{\partial}{\partial x^{a_3}} \circ \iota \]
where \( dt(\frac{\partial}{\partial x^{a_i}}) = \frac{\partial}{\partial x^{a_i}} \circ \iota, \ dt(\frac{\partial}{\partial x^{a_i}}) = (\frac{\partial}{\partial x^{a_1}} + \frac{\partial}{\partial x^{a_2}}) \circ \iota \) and \( dt(\frac{\partial}{\partial x^{a_i}}) = \frac{\partial}{\partial x^{a_i}} \circ \iota. \) Hence for the vector field
\[ 2X = (\xi^{a_i} \circ \iota) \frac{\partial}{\partial x^{a_i}} + \frac{1}{2} (\xi^{a_i} \circ \iota + \xi^{a_i} \circ \iota) (\frac{\partial}{\partial x^{a_1}} + \frac{\partial}{\partial x^{a_2}}) + (\xi^{a_3} \circ \iota) \frac{\partial}{\partial x^{a_3}} \]
on \( 2M \), we have
\[ \overline{X} \big|_{2M} = dt(2X) + \frac{1}{2} (\xi^{a_i} - \xi^{a_i}) \circ \iota \cdot (\frac{\partial}{\partial x^{a_1}} - \frac{\partial}{\partial x^{a_2}}) \circ \iota \] (1)