SMALL COMPLETE CAPS IN $PG(2,q)$, FOR $q$ AN ODD SQUARE

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An interesting class of $k$-arcs with $k = 4(\sqrt{q} - 1)$ in the projective plane over $GF(q)$ is constructed for $q$ an odd square; the construction yields many complete arcs of small size in $PG(2,q)$ when $q < 2401$.

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1 INTRODUCTION

In $PG(2,q)$ a $k$-arc with $k \leq \frac{1}{2}(3+\sqrt{8q+1})$ is never complete (see [2]) and the values of $k$ for the very few complete $k$-arcs explicitly constructed up to now far exceed the above bound (see [3], [4], [5]). In this paper, a class of $4(\sqrt{q} - 1)$-arcs is constructed in $PG(2,q)$, when $q = m^2$, $m = p^k$, $p$ odd prime; for $q \leq 2401$ some of these arcs are complete, and for $q = 81, 169, 289, 625, 729, 841, 1369, 1681, 2401$ their size is the lowest compared to the complete arcs known up to now. So they provide examples of 1-error correcting quasi-perfect linear codes over $GF(q)$ whose length is small compared to $q$. The idea which the construction is based on is the same as in [1], and here it is applied to a more general case. The arcs are constructed by using two conics: half of their points lie on one conic, the other half on the second conic. The steps followed and notation used are very similar to those in [1]. More precisely [1] provides one $4(\sqrt{q} - 1)$-arc in $PG(2,q)$ when $q = p^2$ and $p$ is a prime, $p \equiv 3$ (mod 4); here several $4(\sqrt{q} - 1)$-arcs are defined in $PG(2,q)$ when $q$ is an odd square.

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2 THE CONSTRUCTION OF THE ARCS

Throughout this paper, $p$ is an odd prime, $m = p^h$, $q = m^2$, $F = GF(m)$. Let $\theta$ be a non-square in $F^* := F \setminus \{0\}$; then the polynomial $x^2 - \theta$ is irreducible in $F[x]$ and so the generic element of $GF(q)$ can be written as $a + \beta b$, where $a, b \in F$ and $\beta^2 = \theta$. Consider the following subsets of points in the affine plane $AG(2, q)$:

$$K_1 = \{((\alpha, -\frac{\theta}{\alpha})| \alpha \in F^*\}; \quad K_2 = \{((\beta, -\frac{\theta \beta}{\beta})| \beta \in F^*\};$$

$$K_3 = \{(\gamma, -\frac{\theta}{\gamma})| \gamma \in F^*\}; \quad K_4 = \{(\delta, -\frac{1}{\delta})| \delta \in F^*\};$$

Let $K(\theta) = K_1 \cup K_2 \cup K_3 \cup K_4$. Obviously, $K(\theta)$ can be considered also as a subset of $PG(2, q)$. Observe that the points of $K_1 \cup K_4$ lie on the conic with equation $xy = -\theta$, while those of $K_2 \cup K_3$ lie on the conic with equation $xy = -\theta\beta$.

**Definition 2.1** Let $\varphi_\rho$, $\psi$, and $\eta$ be collineations of $AG(2, q)$ defined by

$$\varphi_\rho(x, y) = (\rho x, \frac{y}{\rho}), \forall \rho \in F^*; \quad \psi(x, y) = (y, x); \quad \eta(x, y) = (\frac{x}{y}, y).$$

These collineations (and so all the collineations of the group that they generate) fix $K(\theta)$. They act in the following way:

$$\varphi_\rho: K_i \rightarrow K_i, \quad i \in \{1, 2, 3, 4\};$$
$$\psi: K_2 \leftrightarrow K_3, \quad K_1 \rightarrow K_1, \quad K_4 \rightarrow K_4;$$
$$\eta: K_2 \leftrightarrow K_3, \quad K_1 \leftrightarrow K_4;$$
$$\eta \psi: K_1 \leftrightarrow K_4, \quad K_2 \rightarrow K_2, \quad K_3 \rightarrow K_3.$$

**Proposition 2.1** $K(\theta)$ is a $4(\sqrt{q} - 1)$-arc in $PG(2, q)$.

**Proof.** Let $P_1$, $P_2$, $P_3$ be distinct points of $K(\theta)$. The possibilities are the following:

- $P_1, P_2, P_3$ in $K_1 \cup K_4$ or in $K_2 \cup K_3$: in either case these points are not collinear as they belong to an irreducible conic.

- $P_1 \in K_1, P_2 \in K_1, P_3 \in K_2$;
  Let $P_1 = (\alpha, -\theta/\alpha), P_2 = (\beta, -\theta/\beta), P_3 = (\gamma, -\theta/\gamma)$ with $\alpha, \beta, \gamma \neq 0, \alpha \neq \beta$. Suppose without loss of generality that $\alpha = 1$ (if $\alpha \neq 1$ use $\varphi_{1/\alpha}$). Suppose these points are collinear; so

$$\begin{vmatrix}
1 - \beta & 1 - \gamma \\
-\theta + \theta/\beta & -\theta + \theta/\gamma
\end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix}
1 - \beta & 1 - \gamma \\
1 - 1/\beta & 1 - 1/\gamma
\end{vmatrix} = 0$$

$$\Leftrightarrow \frac{(1 - \beta)(\gamma - \beta)}{\gamma} = \frac{(1 - \gamma)(\beta - 1)}{\beta}.$$