On critical circle homeomorphisms
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— Dedicated to the memory of R. Mañé

Abstract. We prove that an analytic circle homeomorphism without periodic orbits is conjugated to the linear rotation by a quasi-symmetric map if and only if its rotation number is of constant type. Next, we consider automorphisms of quasi-conformal Jordan curves, without periodic orbits and holomorphic in a neighborhood. We prove a "Denjoy theorem" that such maps are conjugated to a rotation on the circle.

Keywords: Cross-ratio inequality, quasi-symmetric conjugacy, Denjoy theorem.

1. Circle Maps
1.1. Introduction

Apparently due to the great complexity of problems involved, there is a tendency to divide dynamical systems into ever smaller sub-fields, each developing in its own right. Only exceptional mathematicians are able to overcome this tendency. Ricardo Mañé was this type of researcher. He left a deep mark on ergodic theory, general theory of diffeomorphisms on surfaces, hamiltonian dynamics, and the systems in one real or complex dimension. To him, dynamical systems was one field.

This paper is not directly connected to Mañé’s research. Without trying to approach his scope of vision, we show another example of a close connection between the theory in one real and complex dimension. Both sub-fields are quite different in the tools they use: strong reliance on the ordering of points in real dynamics and methods of complex function theory in holomorphic dynamics. Interactions between these theories have repeatedly been shown to lead to new deep results. An

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example of this occurred in 1987 when Michel Herman proved a theorem about boundaries of Siegel disks with rotation numbers of constant type. His solution was based on quasiconformal surgery, see [4], and a theorem about the quasi-symmetric conjugacy between an analytic homeomorphism of the circle and a linear rotation, see [3].

In the first section we provide a proof of Herman's theorem about circle mappings. Our proof is somewhat different from the original one and based on the method of [2]. We introduce a new technical concept of a cross-ratio module as the "minimal" tool which makes the theory work. Another purpose of Section 1 is to simply fill out a gap in the literature by publishing proofs of results which have been referenced several times.

The second section contains a generalization of the theorem by Yoccoz concerning the absence of Denjoy counterexamples for analytic maps. We extend the result to holomorphic maps that preserve an arbitrary quasiconformal Jordan curve. This part of our work provides a good example of the interplay between "real" arguments based on the ordering of points and complex function-theoretic tools.

Quasi-symmetric homeomorphisms of the line.

**Definition 1.1.** A homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) is called \( Q \)-quasi-symmetric if and only if for every real \( x \) and \( \delta \neq 0 \)
\[
\left| h(x + \delta) - h(x) \right| \leq Q, \quad \left| h(x) - h(x - \delta) \right| \leq Q.
\]

**Rotation numbers.** If \( f : \mathbb{R} \to \mathbb{R} \) is the lifting of a degree 1 homeomorphism of the circle, i.e. \( f(x + 1) = f(x) + 1 \) and \( f \) is increasing, then the following limit:
\[
\rho(f) := \lim_{n \to \infty} \frac{f^n(x)}{n}
\]
exists for every \( x \) and is independent of \( x \). It is called the rotation number of \( f \). A lifting can always be chosen so that \( \rho(f) \in [0, 1) \), so we assume that in the sequel. An irrational rotation number can be