NUMERICAL DETERMINATION OF LISSAJOUS TRAJECTORIES IN THE RESTRICTED THREE-BODY PROBLEM

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Abstract. In previous studies, Lissajous trajectories associated with the collinear libration points in the restricted three-body problem have been successfully computed analytically to at least third-order. Those approximations are utilized to determine such trajectories numerically for an arbitrary, predetermined number of revolutions in the rotating frame, for the case of circular primary motion. The numerical approach first identifies target positions at specified intervals along the trajectory and locates a continuous path through those points with velocity discontinuities. Then the $\Delta v$’s are simultaneously reduced in an iterative process. Such trajectories have been constructed in various primary systems, for a wide range of orbit sizes and a large number of revolutions.

1. Introduction

In the restricted three-body problem a particular type of bounded, three-dimensional solution has been recently studied by a number of authors. These trajectories are generally quasi-periodic and associated with each of the collinear libration points. Farquhar and Kamel [1] used the method of Linstedt–Poincaré to produce a third-order analytic solution for such orbits near the translunar libration point ($L_2$) in the Earth–Moon system. Richardson and Cary [2] also developed a series solution truncated to fourth order. The Lindstedt method has also been used successfully to investigate three-dimensional orbits in other dynamical systems [3]. These analytic approaches show that the linearized motion near any collinear point includes a periodic path in the plane of primary motion, and an uncoupled periodic out-of-plane motion. The two frequencies are generally unequal. For small amplitudes, the orbital path traces out a Lissajous figure, and thus, the orbits will subsequently be called Lissajous trajectories. When the amplitude is sufficiently large, so that nonlinear terms are significant, certain combinations of in-plane and out-of-plane amplitudes exist such that the corresponding frequencies are equal and a perfectly periodic three-dimensional motion results. Members of this subset of general Lissajous trajectories are sometimes called halo orbits.

Halo orbits have been computed analytically using the series solutions mentioned above [4]. In addition, they have been calculated numerically from the exact nonlinear equations of motion using differential corrections schemes, the results of which appear in references [5] through [8] among others. Lissajous trajectories have been computed analytically from the series solutions but numeric calculation has been limited because of their nonperiodicity. The objective of this work was to produce a continuous, bounded, ‘Lissajous’ solution numerically from the nonlinear differential equations. The original motivation for this study was actually to
determine a nominal Lissajous trajectory to be associated with the interior collinear libration point \(L_1\) in the Sun–Earth system. Therefore, the constants corresponding to that problem were used primarily. The results, however, are easily generalized to other systems.

In this initial effort, it was assumed that the primaries move in circular orbits. Also, in development of a technique to meet the objective, certain abilities of the resulting method were defined as desirable. Lissajous trajectories exhibit a far greater variety of orbit sizes than halo orbits. The numeric approach should accommodate as much of that range as possible. The approach also needed to be able to find a solution for a possibly predetermined but arbitrary number of revolutions in the rotating frame. At the same time, numerical problems inherent with long integration times had to be minimized.

2. Analysis

2.1. EQUATIONS OF MOTION

The equations governing motion in this problem are written in the form associated with the restricted three-body problem. In the usual rotating coordinate system, the \(x\)-axis is always directed from the larger toward the smaller primary. The \(y\)-axis is \(90^\circ\) from the \(x\)-axis in the primary plane of motion. The \(z\)-axis completes the right handed system, defining the out-of-plane direction. The associated unit vectors are \(\hat{x}, \hat{y}, \hat{z}\) respectively. The problem is nondimensionalized such that the following quantities are equal to unity: the sum of the masses of the primaries, the mean distance between them, the mean angular velocity of the coordinate frame, and the gravitational constant. The nondimensional smaller primary mass is represented as \(\mu\).

Let the vector \(\vec{p}\) describe the position of the infinitesimal mass from the center of mass of the primaries such that \(\vec{p}\) has components \(x, y, z\). In the standard formulation of the circular restricted three-body problem, then, the equations of motion can be written

\[
\begin{align*}
\dot{x} - 2\dot{y} &= \frac{\partial U}{\partial x} \\
\dot{y} + 2\dot{x} &= \frac{\partial U}{\partial y} \\
\dot{z} &= \frac{\partial U}{\partial z},
\end{align*}
\]

where

\[
\begin{align*}
U &= (x^2 + y^2)/2 + (1 - \mu)/d + \mu/r \\
d &= \left[ (x + \mu)^2 + y^2 + z^2 \right]^{1/2} \\
r &= \left[ (x - 1 + \mu)^2 + y^2 + z^2 \right]^{1/2}.
\end{align*}
\]