HAMILTON-LIKE VECTORS FOR A CLASS OF KEPLER PROBLEMS WITH A FORCE PROPORTIONAL TO THE VELOCITY

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Abstract. The Kepler problem for the resistive force $\alpha/r^2$ is known to have a conserved vector which is the analogue to Hamilton’s vector for the standard Kepler problem. In this note it is shown in a very elementary way that many similar force laws display the same property. The orbit equation can be obtained easily in such cases.

1. Introduction

The Kepler problem for resistive forces proportional to the velocity is of interest in the study of the motion of low altitude bodies which are affected by the upper regions of the Earth’s atmosphere. This was a problem for some early artificial satellites. In more recent years the intended or unintended re-entry of orbiting craft into the Earth’s atmosphere has become a matter of practical importance. Few solutions to this type of Kepler problem, either analytic or in closed form, exist.

Brouwer and Hori (1961) obtained a closed form solution which included first order corrections due to a velocity square law in drag acceleration. (It is generally accepted that a drag force is proportional to the square of the velocity and to an atmospheric density such as $\alpha/\rho^2$.) Danby (1962) proposed a ‘drag-like’ force law proportional to the velocity and inversely proportional to the square of the radial distance. For a small constant of proportionality he obtained a first order perturbation solution. Subsequently Mittleman and Jezewski (1982), Jezewski and Mittleman (1983) and Leach (1986) have shown that there exist first integrals which are the direct analogues to the angular momentum, the energy and the Laplace–Runge–Lenz vector of the classical Kepler problem.

In this note we show that there are many resistive force laws for which similar results hold. Some of the possible forces could not be considered physical, but the extension of the class of forces for which closed form results are available has an intrinsic interest and may be of value. In common with Collinson (1973), Sarlet and Bahar (1980), Gorringe and Leach (1986) and Leach (1986) our approach is very simple. We manipulate the equation of motion to recast it into forms which lead naturally and readily to first integrals.

In general the result may be expressed in closed form rather than analytically. This is already the case for the force law mentioned above as the conserved vector which is the analogue to the Laplace–Runge–Lenz vector can be expressed in terms of the standard sine and cosine integrals. However, this causes no difficulties in the computation of the orbit.

Although the method which we use here is rather simple and specific, it does have

the advantage of preserving physical intuition. One only has to compare the simplicity of the work of Gorringe and Leach (1986) on the ‘time-dependent Kepler problem’ with equation of motion

$$\ddot{r} = \ddot{u}(t)r/\dot{u}(t) - \mu r/(r^3 \dot{u}(t))$$

(1)

with the complexity of previous work by Katzin and Levine (1983) and Leach (1985) on the same problem. This present work shows that it can produce further results and extension to a wider class of problems is envisaged.

2. Statement of the Problem and a First Integral Related to Angular Momentum

The problem which we study is the Kepler problem with an additional force which is proportional to the velocity. The equation of motion is

$$\ddot{r} + f \dot{r} + \mu r/r^3 = 0$$

(2)

in reduced coordinates where $\mu$ is a constant and the functional dependence of the scalar function $f$ is as yet unspecified. (One could rescale (2) to remove $\mu$, but it is kept for physical considerations.) Taking the vector product of (2) with $r$ we have

$$r \times \ddot{r} + f \dot{r} \times \dot{r} = 0$$

(3)

where $L = r \times \dot{r}$ is the angular momentum. Taking the vector product of $L$ with (3),

$$L \times \ddot{L} = 0$$

(4)

so that, although the angular momentum is not conserved, its direction is. The motion lies in a fixed plane and we take the origin to lie within the plane so that, when we require a co-ordinate representation, we may use plane polar co-ordinates $(r, \theta)$. Rewriting (3) as

$$(L + fL)L = 0$$

(5)

we have $L$ and $f$ related by

$$L/L = -f.$$ 

(6)

We may obtain a first integral related to the angular momentum if it is possible to integrate (6). For example, with the drag-like force law proposed by Danby (1962),

$$f = \alpha/r^2$$

(7)

and (6) becomes

$$L + fL = \dot{L} + (\alpha/r^2)r^2 \theta$$

$$= \dot{L} + \alpha \theta$$

(8)

which gives the first integral

$$h = L + \alpha \theta.$$ 

(9)