A SPLINE COLLOCATION METHOD FOR SINGULAR INTEGRAL EQUATIONS WITH PIECEWISE CONTINUOUS COEFFICIENTS

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We consider the collocation method with piecewise linear trial functions for systems of singular integral equations with Cauchy kernel and piecewise continuous coefficients. Necessary and sufficient conditions for the stability in $L^2$ are given. The results are obtained in the case of a closed Ljapunov curve as well as in the case of an interval. The proof of the main theorem is based on a modification of the Banach algebra technique established in the local principle by Gohberg and Krupnik [2]. Our results extend those obtained by Prößdorf and Schmidt [9,10] from the case of continuous coefficients and unit circle to the case of piecewise continuous coefficients.

0. INTRODUCTION

Let $\Gamma$ be a simple closed Ljapunov curve in the complex plane $\mathbb{C}$ given by a regular parametric representation

$$
\Gamma: t = \gamma(s) = \gamma_1(s) + i\gamma_2(s), \quad s \in \mathbb{R},
$$

where $\gamma$ is a 1-periodic function of the real variable $s$, $|\gamma'(s)| = |dt/ds| \neq 0$ and

$$
|\gamma'(s) - \gamma'(\sigma)| \leq C|s - \sigma|^{\alpha}, \quad s, \sigma \in \mathbb{R}
$$

with a positive constant $C$ and $0 < \alpha \leq 1$.

We next introduce several notational conventions: $L^2(\Gamma)$ is the Hilbert space of all square Lebesgue integrable (complexvalued) functions on $\Gamma$ with scalar product

$$
(f_1, g_1) := \int_{\Gamma} f_1(t)\overline{g_1(t)}|dt|; \quad f_1, g_1 \in L^2(\Gamma).
$$

Further, $R(\Gamma)$ stands for the Banach space of all bounded and Riemann integrable functions on $\Gamma$ with norm
\[ \|f\|_\infty := \sup_{t \in T} |f(t)|. \] By \( C(T) \) (\( CR(T) \)) we denote the Banach space of continuous functions on \( T \). The symbol \( PC(T) \) designates the algebra of functions on \( T \) which are piecewise continuous as well as continuous from the right on \( T \) in the following sense: For all \( t = \gamma(s) \in T \), the limits
\[ f(t \pm 0) := \lim_{\sigma \to s^\pm} f(\gamma(\sigma)) \]
exist and are finite, \( f(t+0) = f(t) \) and \( f \) is discontinuous at most at a finite number of points \( t \in T \). Let \( \overline{PC}(T) \) be the closure of the algebra \( PC(T) \) with respect to the norm \( \| \cdot \|_\infty \).

If \( E(T) \) is any of the spaces mentioned above, then by \( E_m(T) \) (\( m \) a natural number) we denote the Banach space of all \( m \)-dimensional vectors \( f = (f_1, \ldots, f_m) \) with components \( f_j \in E(T) \) \( (j = 1, \ldots, m) \). The norm of the vector \( f \) is defined as the sum of the norms of the components \( f_j \). By \( E_{mxm}(T) \) we denote the set of all \((mxm)\)-matrices with elements from \( E(T) \). Accordingly, \( L_m^2(T) \) is a Hilbert space with scalar product
\[ (f, g) := \int_T [f(t), g(t)] dt; \quad f, g \in L_m^2(T), \]
where \([\cdot, \cdot]\) stands for the scalar product in the Euclidian space \( C_m \); the norm in \( C_m \) will be denoted by \( |\cdot| \). Finally, by \( S_T \) we denote the Cauchy operator of singular integration on \( T \)
\[ (S_T f)(t) := \frac{1}{\pi i} \int_T \frac{f(\tau)}{\tau - t} d\tau \quad (t \in T), \]
which is a bounded linear operator in \( L_m^2(T) \). The associated projections are \( P_T := \frac{1}{2}(I+S_T) \) and \( Q_T := I-P_T \).

We shall now consider the singular integral equation of the form
\[ (Ax)(t) := a(t)x(t) + b(t)(S_T x)(t) = y(t) \quad (1) \]
with coefficients \( a, b \in \overline{PC}_{mxm}(T) \) in \( L_m^2(T) \). The object which we seek is an approximate solution of Equation (1) in form of an \( m \)-vector valued piecewise linear function
\[ x_n(t) = \sum_{k=0}^{n-1} \xi_k \phi_k(t) \]
with unknown coefficients \( \xi_k = \xi_k^{(n)} \in C_m \) and piecewise linear