A SPLINE COLLOCATION METHOD FOR SINGULAR INTEGRAL EQUATIONS WITH PIECEWISE CONTINUOUS COEFFICIENTS

Siegfried Prößdorf and Andreas Rathsfeld

We consider the collocation method with piecewise linear trial functions for systems of singular integral equations with Cauchy kernel and piecewise continuous coefficients. Necessary and sufficient conditions for the stability in $L^2$ are given. The results are obtained in the case of a closed Lyapunov curve as well as in the case of an interval. The proof of the main theorem is based on a modification of the Banach algebra technique established in the local principle by Gohberg and Krupnik [2]. Our results extend those obtained by Prößdorf and Schmidt [9,10] from the case of continuous coefficients and unit circle to the case of piecewise continuous coefficients.

0. INTRODUCTION

Let $\Gamma$ be a simple closed Lyapunov curve in the complex plane $\mathbb{C}$ given by a regular parametric representation

$$\Gamma : t = \gamma(s) = \gamma_1(s) + i\gamma_2(s), \quad s \in \mathbb{R},$$

where $\gamma$ is a 1-periodic function of the real variable $s$, $|\gamma'(s)| = |dt/ds| \neq 0$ and

$$|\gamma'(s) - \gamma'(s')| \leq C|s - s'|^\alpha, \quad s, s' \in \mathbb{R}$$

with a positive constant $C$ and $0 < \alpha \leq 1$.

We next introduce several notational conventions: $L^2(\Gamma)$ is the Hilbert space of all square Lebesgue integrable (complex-valued) functions on $\Gamma$ with scalar product

$$(f_1, g_1) := \int_\Gamma f_1(t)\overline{g_1(t)}|dt|; \quad f_1, g_1 \in L^2(\Gamma).$$

Further, $R(\Gamma)$ stands for the Banach space of all bounded and Riemann integrable functions on $\Gamma$ with norm
\[ \|f\|_\infty := \sup_{t \in \Gamma} |f(t)|. \]

By \( C(\Gamma) \) (\( C^r(\Gamma) \)) we denote the Banach space of continuous functions on \( \Gamma \). The symbol \( PC(\Gamma) \) designates the algebra of functions on \( \Gamma \) which are piecewise continuous as well as continuous from the right on \( \Gamma \) in the following sense: For all \( t = \gamma(s) \in \Gamma \), the limits

\[ f(t \pm 0) := \lim_{\sigma \to s \pm 0} f(\gamma(\sigma)) \]

exist and are finite, \( f(t+0) = f(t) \) and \( f \) is discontinuous at most at a finite number of points \( t \in \Gamma \). Let \( \overline{PC}(\Gamma) \) be the closure of the algebra \( PC(\Gamma) \) with respect to the norm \( \| \cdot \|_\infty \).

If \( E(\Gamma) \) is any of the spaces mentioned above, then by \( E_m(\Gamma) \) (\( m \) a natural number) we denote the Banach space of all \( m \)-dimensional vectors \( f = (f_1, \ldots, f_m) \) with components \( f_j \in E(\Gamma) \) (\( j = 1, \ldots, m \)). The norm of the vector \( f \) is defined as the sum of the norms of the components \( f_j \). By \( E_{m \times m}(\Gamma) \) we denote the set of all \((m \times m)\)-matrices with elements from \( E(\Gamma) \). Accordingly, \( L_m^2(\Gamma) \) is a Hilbert space with scalar product

\[ (f, g) := \int_\Gamma [f(t), g(t)] \, dt \quad f, g \in L_m^2(\Gamma), \]

where \([\ldots, \ldots]\) stands for the scalar product in the Euclidean space \( C_m^2 \); the norm in \( C_m^2 \) will be denoted by \( |\cdot| \). Finally, by \( S_\Gamma \) we denote the Cauchy operator of singular integration on \( \Gamma \)

\[ (S_\Gamma f)(t) := \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} \, d\tau \quad (t \in \Gamma), \]

which is a bounded linear operator in \( L_m^2(\Gamma) \). The associated projections are \( P_\Gamma := \frac{1}{2} (I + S_\Gamma) \) and \( Q_\Gamma := \frac{1}{2} (I - S_\Gamma) \).

We shall now consider the singular integral equation of the form

\[ (Ax)(t) := a(t)x(t) + b(t)(S_\Gamma x)(t) = y(t) \quad (1) \]

with coefficients \( a, b \in \overline{PC}_{m \times m}(\Gamma) \) in \( L_m^2(\Gamma) \). The object which we seek is an approximate solution of Equation (1) in form of an \( m \)-vector valued piecewise linear function

\[ x_n(t) = \sum_{k=0}^{n-1} \xi_k \phi_k^{(n)}(t) \]

with unknown coefficients \( \xi_k = \xi_k^{(n)} \in C_m \) and piecewise linear