SOLUTION OF THE THREE-DIMENSIONAL INVERSE PROBLEM

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Abstract. The three dimensional inverse problem for a material point of unit mass, moving in an autonomous conservative field, is solved. Given a two-parametric family of space curves \( f(x,y,z) = c_1 \), \( g(x,y,z) = c_2 \), it is shown that, in general, no potential \( U = U(x,y,z) \) exists which can give rise to this family. However, if the given functions \( f(x,y,z) \) and \( g(x,y,z) \) satisfy certain conditions, the corresponding potential \( U(x,y,z) \), as well as the total energy \( E = E(f,g) \) are determined uniquely, apart from a multiplicative and an additive constant.

1. Introduction

In 1974 V. Szebehely derived a first order linear partial differential equation for the potential \( U = U(x,y) \) generating a prescribed monoparametric family of planar curves \( f(x,y) = C \), traced by a unit mass \( m = 1 \), with any desired total energy dependence \( E = E(f) \). Since then, a number of papers have appeared, dealing with various aspects of this version of the inverse problem. To quote some of these papers, we mention Broucke and Lass (1977), Morrison (1977), Broucke (1979), Szebehely and Broucke (1981), Molnár (1981), Mertens (1981), Melis and Piras (1982), Bozis (1983a, b), Xanthopoulos and Bozis (1983), Bozis (1984), Puel (1984), Bozis and Mertens (1985).

Very recent work on the inverse problem mostly refers to generalization of Szebehely’s equation to three dimensions.

Bozis (1983c) extended Dainelli’s formulae, as given by Whittaker (1944), to three dimensions and found two equations relating the force components \( X(x,y,z), Y(x,y,z), Z(z,y,z) \) of an autonomous (not necessarily conservative) field to the functions \( f \) and \( g \) of the two-parametric family of orbits \( f(x,y,z) = c_1 \), \( g(x,y,z) = c_2 \). A similar extension of Dainelli’s formulae in the \( n \)-dimensional space \( \mathbb{R}^n \) \( (n \geq 2) \) has been presented recently by Gascon, Lopez and Broncano (1984). The above two papers contain comments as to how the formulae for the force components \( X, Y, Z \) reduce to Érdi’s (1982) equations in the case that the dynamical system is assumed to be conservative. A paper by Melis and Piras (1984) examines the problem of the determination of the potential \( U = U(q_1, q_2, \ldots, q_n) \) which generates a given family of ‘orbits’ \( f_i(q_1, q_2, \ldots, q_n) = c_i \), for \( i = 1, 2, \ldots, n-1 \), in the \( n \)-dimensional configuration space of the representative point \( (q_1, q_2, \ldots, q_n) \) of a holonomic system; it is found that the potential must satisfy \( (n-1) \) equations of the Szebehely type.

Érdi (1982) was the first author to give a system of two first order, linear partial
differential equations for the determination of the potential \(U = U(x, y, z)\) producing
a prescribed monoparametric family of curves \(f(x, y) = c, z = g(x, y)\). Érdi's equations
are of the Szébehely type and they include the total energy which is assumed to be
given.

A very crucial question regarding Érdi's two equations was put and answered the
following year by Váradi and Érdi (1983). For a two-parametric family of curves
\(f(x, y, z, c_1, c_2) = 0, g(x, y, z, c_1, c_2) = 0\) and for a given energy dependence
\(E = E(c_1, c_2)\) are Érdi's equations consistent? By considering the energy to be
known \textit{a priori} on each prescribed orbit of the given family, the authors established
a necessary condition for the two equations to be consistent and concluded that (i)
in general, a unique solution for the potential does exist; (ii) in special cases more
than one solution exists and (iii) in other (yet degenerate) cases no solution exists.

Evidently, the problem of the existence of a potential for the non-planar case
\((n \geq 3)\) is crucial. Any system, of the sort that we mentioned above, of partial
differential equations in \(U(x, y, z)\) would be of limited importance (if not
meaningless) if necessary and/or sufficient conditions are not available for the system
to admit of a solution.

The present paper is closely related to the work by Váradi and Érdi (1983). In fact
our conclusion regarding the existence of a solution is different from theirs and this
must be attributed to the fact that we formulate the problem on a different prospect.
Our question is: Given \textit{only} a two-parametric family of space orbits \(f(x, y, z) = c_1, g(x, y, z) = c_2\), is there a potential \(U = U(x, y, z)\) which can give rise to this family? If
the answer to this question is affirmative, what should then be the function giving
the energy dependence \(E = E(f, g)\)?

In what follows we show that both questions are meaningful. The energy
\(E = E(f, g)\) not only needs not but, in fact, \textit{can not} be given \textit{a priori}. In order that a
potential exists it is necessary and sufficient that, in general, the energy \(E\) satisfies a
system (we call it system-\(E\)), of four linear partial differential equations with
coefficients depending merely on the given orbit. Two of these equations are of the
second and two are of the third order. To each solution of system-\(E\) there
corresponds one potential which is found explicitly. Attention therefore is directed
to the solution of this system in which the energy \(E(f, g)\) appears as the only
unknown.

It is shown that system-\(E\) is compatible and can be solved, in principle, if certain
necessary and sufficient conditions on the given orbits are satisfied. Due to the
complexity of the algebra involved we did not write down these conditions explicitly
in terms of the given functions \(f(x, y, z)\) and \(g(x, y, z)\), although, of course, this could
be done in principle. On the other hand, system-\(E\) generally consists of four
equations but, in certain cases (as in the Example treated in Section 3) it may be
reduced to a system of three equations, one of the second and two of the third order.
Accordingly, the number of conditions for system-\(E\) to be compatible may vary.

Our approach differs from that of Váradi and Érdi (1983) in that they consider the