Rings with Minimum Condition on Principal Ideals

By CARL FAITH in University Park, Pennsylvania

The purpose of this note is to add to a preliminary investigation by KAPLANSKY [5, § 8] of rings with minimum condition on principal left ideals.

For convenience below, a list of symbols and terminologies is inserted here:

- **MP-ring** = a ring with minimum condition on principal left ideals.
- **M-ring** = a ring containing a minimal left ideal.
- **M-semi-simple ring** = a direct sum of simple M-rings.
- **m-ring** = a ring with minimum condition on left ideals.
- **m-semi-simple ring** = an m-ring which is a direct sum of simple rings.
- **J(A)** = the Jacobson radical of the ring A.
- **L(A)** = the (Baer) lower nil radical of the ring A.
- **(a)_{i}** = the principal left ideal generated by a \( a \in A \).

We recall that a nil radical of a ring A is an ideal M of A satisfying (1) M is nil and (2) \( A - M \) contains no nilpotent ideals \( \neq \{0\} \); and that \( L(A) \) is a nil radical contained in every nil radical of A. Since the Jacobson radical of a ring contains every nil ideal of that ring, and since \( J(A - J(A)) = \{0\} \) [4, p. 5, Theorem 2.2], it is easy to see that \( J(A) \) is a nil radical if and only if \( J(A) \) is nil. The following lemma, used in the proof of our main result (Theorem 1), gives a condition for this.

**Lemma.** If A is a ring with minimum condition on principal ideals of the form \( (a^i)_L \), \( a \in A \), \( i = 1, 2, \ldots \), then \( J(A) \) is a nil radical of A.

**Proof.** If \( a \in J(A) \), then \( (a^N)_L = (a^{N+1})_L \) for suitable \( N \), \( a^N = ba^{N+1} + qa^{N+1} \), where \( q \) is an integer, and \( b \in A \), that is, \( a^N - ca^N = 0 \), where \( c = ba + qa \in J(A) \). Then

\[
a^N = a^N - (c + c' - c')a^N = (a^N - ca^N) - c'(a^N - ca^N) = 0,
\]

where \( c' \) is the quasi-inverse of \( c \). Since \( J(A) \) is nil, \( J(A) \) is nil radical 2).

The lemma shows in an MP-ring A that \( J(A) \supseteq L(A) \). Our main result establishes the reverse inclusion.

**Theorem 1.** If A is an MP-ring, then \( J(A) = L(A) \).

**Proof.** We need the following facts noted by KAPLANSKY in the proof of [5, Theorem 8.1]: (I) Every homomorphic image of an MP-ring is an MP-ring; (II) Every MP-

1) By convention, \( A^2 = \{0\} \) is ruled out when \( A \) is simple.
2) Thus, \( J(A) \) is the maximal nil ideal of \( A \). Cf. LEVITZKI [7, p. 226, Theorem 5.13].
ring is an $M$-ring. LEVITZKI [6, p. 28, Theorem 2] has shown that $L(A)$ coincides with the McCoy radical of $A$, and, as a consequence [6, Corollary 2], established that $A - L(A)$ is a subdirect sum of prime rings $\{P_x | x \in A\}$. Since $A$ is homomorphic to $P_x$, $A \sim A - L(A) \sim P_x$, each $P_x$ is an $MP$-ring by (I), and, hence, an $M$-ring by (II). By McCoy’s [8, p. 831, Theorem 7] every prime $M$-ring is primitive. It follows, since $A - L(A)$ is now a subdirect sum of primitive rings, that $J(A - L(A)) = \{0\}$ [4, p. 5, Theorem 2.2]. Since $J(A - L(A)) \supseteq J(A) - L(A)$, evidently $J(A) = L(A)$ as required.

Kaplansky’s [5, p. 74, Theorem 8.1] implies that every $MP$-ring with $J(A) = \{0\}$ is $M$-semisimple. In view of Theorem 1, every $MP$-ring with $L(A) = \{0\}$ is also $M$-semisimple. This establishes the sufficiency of the next theorem, since the equivalence of (I) with the vanishing of $L(A)$ is well known.

**Theorem 2.** $A$ is $M$-semisimple if and only if: (1) $A$ contains no nilpotent ideals $\neq \{0\}$ and (2) $A$ is an $MP$-ring.

**Proof.** The necessity remains. Since $A$ is $M$-semisimple, $A$ is a (group theoretical) direct sum of minimal left ideals $\{L_x | x \in A\}$, since each simple $M$-ring is (Dieudonné [3, p. 52, Proposition 1]). Thus, each $e \in A$ has a unique representation

$$ (*) \quad e = e_{a_1} + \ldots + e_{a_k}, $$

where $e_{a_j} \in L_{a_j}$, and the $L_{a_j}$, $j = 1, \ldots, k$, are distinct. Let $(e)L$ be a principal left ideal of $A$, where $e \in A$ is represented by $(*)$. If $a \in A$, then $ae = \sum_{j=1}^{k} a e_{a_j} \in L_{a_1} + \ldots + L_{a_k}$, so that $(e)L \subseteq L_{a_1} + \ldots + L_{a_k}$. The left ideal $L^* = L_{a_1} + \ldots + L_{a_k}$, being a sum of finitely many minimal left ideals of $A$, has minimum condition on left ideals (and not just principal left ideals) of $A$ contained in $L^*$. This shows that every chain

$$ (e)L \supseteq (e_2)L \supseteq (e_3)L \supseteq \ldots $$

is finite, that is, $A$ is an $MP$-ring. Trivially, any $M$-semisimple ring $A$ satisfies (1).

Even though the two are logically equivalent, Theorem 2 has an advantage over Kaplansky’s theorem inasmuch as $J(A) = \{0\}$ implies (1) for arbitrary $A$ but not conversely.

Theorem 2 should be compared with the

**Wedderburn-Artin structure theorem.** A ring $A$ is $m$-semisimple if and only if (1) and (2') $A$ is an $m$-ring.

Kaplansky’s theorem achieves two changes of considerable importance:

(a) Simple $M$-rings replace simple $m$-rings as basic units of structure.

(b) Arbitrarily many direct summands of the basic units of structure replace only finitely many of these.

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3) A more general statement: If $M$ is an $A$-module, and if $M$ is a sum of finitely many $A$-submodules each of which has minimum condition on $A$-submodules, then $M$ has this property. (Cf., e.g., [2, p. 22 Proposition 2 and Corollaire] and [1, p. 14, Corollary 2.2B]).

4) Cf. [1, p. 27, Main Theorem].