Invent. math. 106, 335–388 (1991)

Inventiones mathematicae

Modules over regular algebras of dimension 3

M. Artin[1], **J. Tate**[2], **and M. Van den Bergh**[3]

[1] MIT, Department of Mathematics, Cambridge, MA 02139, USA
[2] University of Texas, Austin, TX 78712, USA
[3] Universitaire Instelling Antwerpen, B-2610 Wilrijk, Belgium

Oblatum 15-VII-1990

Table of contents

1 Introduction 335
2 Modules over regular algebras 337
3 Proof that regular algebras of dimension at most 4 are domains 347
4 Dimensions of the dual modules 352
5 Some more preliminary considerations 355
6 Line modules and their relation to modules of gk-dimension 1 360
7 Characterization of the algebras which are finite modules over their centers 370
8 Twisting a graded algebra by an automorphism, and determination of the algebras associated to non-reduced divisors 378

1 Introduction

Let k be a field. In a previous paper [ATV] (see also [OF]) some graded k-algebras A, regular algebras of dimension 3, were constructed from certain automorphisms σ of elliptic curves or of more general one-dimensional schemes E with arithmetic genus 1, which are embedded as cubics in $\mathbb{P}^2$ or as divisors of bidegree (2, 2) in $\mathbb{P}^1 \times \mathbb{P}^1$. In this correspondence, the points of the scheme E were shown to parametrize certain A-modules called *point modules*. A point module N is a graded right A-module with these properties:

(1.1) (i) $N_0 = k$,
(ii) N_0 generates N, and
(iii) $\dim_k N_i = 1$ for all $n \geqq 0$.

The structure of these point modules is related in a nice way to the geometry of the scheme E and its automorphism σ. For example, if $N = N_p$ is the module corresponding to a point p of E, then the normalized shift N^+, defined by

$$N_i^+ = \begin{cases} N_{i+1} & \text{if } i \geqq 0 \\ 0 & \text{if } i < 0\ , \end{cases} \tag{1.2}$$

is the point module which corresponds to the point σp. The object of this paper is to study point modules and their relation to the geometry of E. The main results were announced in [VdB].

To fix ideas, let us consider the case that our algebra A corresponds to a cubic curve E in the plane. In this case, A is a non-commutative analogue of a polynomial ring in 3 variables. There is a normalizing element g of degree 3 in A which is unique up to constant factor. It is the analogue of the cubic equation defining the curve, and the ring $B = A/gA$ is the analogue of the homogeneous coordinate ring of E, defined explicitly by $B = \oplus H^0(E, \mathscr{L} \otimes \mathscr{L}^\sigma \otimes \ldots \otimes \mathscr{L}^{\sigma^{n-1}})$, where $\mathscr{L} = \mathscr{O}_E(1)$ (see [ATV]).

If R is a graded k-algebra, then by analogy with the commutative case, we imagine Proj R to be defined and to have a geometric meaning, and we think of it as the non-commutative analogue of a projective scheme. Thus Proj A is a non-commutative (or "quantum") analogue of the projective plane $\mathbb{P}^2$. We call two A-modules *equivalent* if they are isomorphic modulo $\mathfrak{m}$-torsion, i.e., if they correspond to the same imagined sheaf on Proj A (see (6.5)).

Again by analogy, if $B = A/gA$ as above, then Proj A contains Proj B as a "closed subscheme". And though the structure of Proj A is somewhat obscure, that of Proj B is well understood. The category of graded left (or of right) B-modules modulo torsion is equivalent to the category of quasi-coherent sheaves on the cubic curve E, just as in the commutative case when σ is the identity (see [AV]). The new feature comes into the shift operation on graded B-modules. In the commutative case, the corresponding operation on sheaves is $\mathscr{F} \rightsquigarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{F}$, where $\mathscr{L} = \mathscr{O}_E(1)$. Here this operation is replaced by the operation $\mathscr{F} \rightsquigarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{F}^\sigma$.

In addition to A and B, we will consider the $\mathbb{Z}$-graded ring $\Lambda = A[g^{-1}]$ obtained by adjoining the inverse of the normalizing element g, and its subring Λ_0 of elements of degree zero. Intuitively, the non-commutative affine scheme Spec Λ_0 plays the role of the "open complement" of Proj B in Proj A. It is clear that the structures of A and of Λ_0 are closely related. For the ring Λ_0, we have the following rather strong dichotomy (see (7.3)).

Theorem I *Let s denote the order of the σ-orbit of the class $[\mathscr{L}]$ of $\mathscr{L} = \mathscr{O}_E(1)$ in the Picard group of E. Then if $s < \infty$, Λ_0 is an Azumaya algebra of rank s^2 over its center, while if $s = \infty$, Λ_0 is a simple ring.*

We are also able to show (7.18) in the elliptic case that if σ itself is of finite order, then some power of the normalizing element g is in the center of A. Using this fact, we derive the result which is one of our main goals (see (7.1)):

Theorem II *A regular algebra of dimension* 3 *is a finite module over its center if and only if the automorphism σ has finite order.*

It is quite easy to exhibit the center of the associated algebra B explicitly, so Theorem II is easy to prove in the linear case [ATV, 8.5]. But since we don't have a conceptual description of the algebra A in terms of its triple $(E, \sigma, \mathscr{L})$ in the elliptic case, we aren't able to exhibit the center of an elliptic algebra A explicitly. Instead, we construct a family of graded A-modules of gk-dimension 1 and fixed multiplicity, such that the intersection of their annihilators is zero. This is the main step, because it proves that A is a polynomial identity ring [SSW].