Continuity of Convolution Semigroups on Hypergroups

Walter R. Bloom and Herbert Heyer

Received October 2, 1987

Let \( K \) be a commutative hypergroup with the property that either the identity character is contained in the support of the Plancherel measure on \( K^\wedge \), or the identity character is not isolated in \( K^\wedge \) and all characters sufficiently close (but not equal) to the identity character vanish at infinity. We present a shift compactness theorem for \( K \) and use this to prove that every symmetric convolution semigroup of probability measures on \( K \) is continuous.

KEY WORDS: Harmonic analysis; probability measure; hypergroup; convolution semigroup; tight; shift compact.

1. INTRODUCTION

It is known (Byczkowski and Zak\(^{(4)}\)) that any symmetric semigroup of probability measures on a locally compact group is continuous. We show that this result holds for a wide range of hypergroups, including compact hypergroups, commutative hypergroups of CI or CII type, and all of the known polynomial hypergroups. Fundamental to this is a sharpening of the shift compactness theorem (Bloom\(^{(1)}\)), which will be the focus of the first half of this article.

Throughout we adopt the notion of a hypergroup \( K \) in the sense of Jewett.\(^{(7)}\) By \( C(K) \), \( C_0(K) \), and \( C_{00}(K) \) we denote the spaces of all bounded functions on \( K \), those that vanish at infinity and those with compact support, respectively. We reserve the symbols \( M(K) \), \( M^+(K) \), and \( M^1(K) \) for the space of all bounded Radon measures, those that are nonnegative and the probability measures on \( K \), respectively. For any \( f \in C(K) \) and

---

1 School of Mathematical and Physical Sciences, Murdoch University, Perth, Western Australia 6150, Australia.
2 Mathematisches Institut, Universität Tübingen, D-7400 Tübingen 1, Federal Republic of Germany.
\( \mu \in M(K) \), \( f^\sim \) and \( \mu^- \) are defined through involution. For a given subset \( A \) of \( K \), \( A^c \) stands for its complement and \( \xi_A \) for its indicator function. Finally \( \varepsilon_x \) denotes the Dirac measure at \( x \in K \).

For the harmonic analysis of hypergroups we refer to Jewett \(^7\) and Bloom and Heyer \(^2,3\). The more straightforward results will be used without special comment. For details of polynomial hypergroups the reader is referred to Lasser \(^8\). These will serve as an important class of examples for the shift compactness results below.

2. SHIFT COMPACTNESS

Important to the study of convolution semigroups of probability measures is the shift compactness theorem, proved for locally compact groups by Parthasarathy, Ranga Rao, and Varadhan \(^10\) which gives various criteria for a net of probability measures to satisfy a tightness condition. Recall that a net \( (\mu_n) \) of probability measures is called uniformly tight if for every \( \varepsilon > 0 \) there exists a compact set \( C \) such that \( \mu_n(C) > 1 - \varepsilon \) for all \( n \), and tight if for every \( \varepsilon > 0 \) there exists a compact set \( C' \) and an index \( n_0 \) such that \( \mu_n(C') > 1 - \varepsilon \) for all \( n \geq n_0 \). The following version of the shift compactness theorem for hypergroups was essentially given in Bloom \(^1\) Theorem 5.2.

**Theorem 1.** Let \( (\mu_n), (v_n), \) and \( (\lambda_n) \) be nets of measures in \( M^1(K) \) with \( \mu_n = v_n \ast \lambda_n \) for all \( n \).

(a) If \( (v_n) \) and \( (\lambda_n) \) are tight then so is \( (\mu_n) \).

(b) If \( (\mu_n) \) and \( (\lambda_n) \) are tight then so is \( (v_n) \).

(c) If \( (\mu_n) \) is tight then there exist \( (x_n), (y_n) \subset K \) such that each of \( (\varepsilon_{x_n} \ast \lambda_n) \) and \( (v_n \ast \varepsilon_{y_n}) \) is tight. If furthermore \( K \) is commutative then \( (x_n), (y_n) \) can be so chosen such that \( (\varepsilon_{y_n} \ast \varepsilon_{x_n}) \) is tight.

The assertions (a)–(c) continue to hold with “tight” replaced by “uniformly tight.”

**Proof.** The proof of (a), (b), and the tightness of \( (\varepsilon_{x_n} \ast \lambda_n) \) in (c) are given in Bloom \(^1\) Theorem 5.2. For the remainder of (c) first note that \( \mu^- = \lambda_n^- \ast v_n^- \) gives the existence of \( (y_n) \subset K \) such that \( (\varepsilon_{y_n} \ast v_n^-) \) is tight, and hence so is \( (v_n \ast \varepsilon_{y_n}) \). We now apply (a) to give that \( (v_n \ast \varepsilon_{y_n} \ast \varepsilon_{x_n} \ast \lambda_n) \) is tight, and then (b) and commutativity to deduce the tightness of \( (\varepsilon_{y_n} \ast \varepsilon_{x_n}) \).

The proof of these assertions with “tight” replaced by “uniformly tight” is analogous. \( \square \)