The Nonemptiness of the $f$-Core of a Game Without Side Payments\footnote{The authors thank two anonymous referees for many helpful comments. The second author is indebted primarily to the Natural Sciences and Engineering Research Council of Canada for financial support and also the Social Sciences and Humanities Research Council of Canada.}

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Abstract: We prove the nonemptiness of the core of a continuum game without side payments where only small coalitions – ones bounded in absolute size of finite cardinality – are permitted. This result covers assignment games with a continuum of players and includes combinations of several assignment games, such as housing and automobile markets.

1 Introduction

In this paper we prove the nonemptiness of the core, called the $f$-core, of a continuum game with finite coalitions. The formulation covers both games with and without side payments. Two conditions are required: (1) the sizes of permissible coalitions must be bounded and (2) Pareto-frontiers for permissible coalitions must have slopes bounded above zero. The second condition is automatically satisfied by games with side payments and excludes, for example, cases where payoff sets consist of isolated points. Since the bound on permissible coalition sizes can be arbitrarily large, the first condition imposes virtually no restriction.

The framework of a game with a continuum of players and finite coalitions, and the concept of the $f$-core, were introduced in Kaneko-Wooders (1986). In the context of an exchange economy with widespread externalities, Hammond-Kaneko-Wooders (1989) proved the equivalence of the $f$-core and competitive outcomes. From this equivalence, together with the existence of a competitive equilibrium, they also obtained the nonemptiness of the $f$-core of an economy. In the same context, Kaneko-Wooders (1989) discussed a finite analogue of the continuum case. In the context of a game without sidepayments, Kaneko-Wooders (1986) demonstrated the nonemptiness of the $f$-core with a finite number of player types. For some applications, the finite types assumption may be cumbersome. This motivates the current paper.
The nonemptiness result of this paper is suitable particularly for economic models with small coalitions. Examples include continuum extensions of the assignment market models of Shapley–Shubik (1971) and Kaneko (1982) and some combinations of these markets, such as housing and automobiles. It is known that a combination of finite assignment markets may lose the nonemptiness of the core. After stating our main result, we illustrate why this emptiness result occurs in a finite world and why nonemptiness is obtained in a continuum world.

Relative to the results of Hammond–Kaneko–Wooders (1989), and Kaneko–Wooders (1989), the essential restriction of the result of the current paper is that the continuum game does not allow widespread externalities. The incorporation of widespread externalities into a continuum game without side payments and the nonemptiness of the $f$-core are open problems.

2 The $f$-Core of a Continuum Game

Let $(N, B, \mu)$ be a measure space, where $N$ is a Borel subset of a complete separable metric space, $B$ is the $\sigma$-algebra of all Borel subsets of $N$, and $\mu$ is a nonatomic measure with $0 < \mu(N) < +\infty$. Each element in $N$ is called a player and $N$ is the player set. The $\sigma$-algebra $B$ is necessary for measurability arguments but does not play a game-theoretic role.

Let $n$, a positive integer, be a bound on coalition sizes. Let $F$ be the set of all finite subsets of $N$ containing no more than $n$ members. Each element $S$ in $F$ is called simply a coalition.

We consider a close of games where the payoffs attainable by a coalition of players depend on the attributes of the members of the coalition. The set of attributes $A$ is given as a compact metric space with metric $d$. Let $A^* = \bigcup_{t=1}^{n} A^t$, where $A^t$ is the $t$-fold Cartesian product of $A$. An element $x$ in $A^t$ is a list of attributes of a $t$-member coalition.

A characteristic function $V^*$ is a function on $A^*$, which assigns to each $t$-vector $x = (x_1, \ldots, x_t) \in A^t$ ($t = 1, \ldots, n$) a nonempty closed subset, $V^*(x)$, of $R^t$ with the following properties:

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2 The mixture of a continuum of players and finite coalitions may raise the question of how we should interpret the individual player relative to the total player set. In the present approach the individual player remains the same as in finite models while the total player set is approximated by a continuum. Mathematically, outcomes of cooperation of finite numbers of players are aggregated into outcomes for the total player set by measurement-consistent partitions, to be defined presently. In the traditional approach to a continuum game, where coalitions are of positive measure, the notion of the individual player becomes vague. For more discussion of these issues, see Kaneko–Wooders (1986, 1989) and Hammond–Kaneko–Wooders (1989).