REMARKS ON THE NUMERICAL INTEGRATION OF
NEAR-PARABOLIC ORBITS

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Abstract. For near-parabolic orbits the distinction between coordinates and elements disappears provided the KS-technique is used. In KS-variables a pure parabolic motion is described by linear functions. Advantage is taken of that fact for establishing numerical procedures in perturbed near-parabolic cases.

Main Considerations

The equations of motion of an artificial satellite are

\[ \ddot{x} + \frac{K^2}{r^3} x = \text{Pert}. \]  

(1)

\( x \) is the position vector in a rectangular frame, \( r \) is the distance from the origin, \( K^2 \) the gravitational parameter. The abbreviation 'Pert' stands for the physical perturbations generated by oblateness, third body, drag and other additional small forces. For sake of step-size adaption we introduce the new independent variable \( s \) which is defined by \( dt = r \, ds \) (\( t \) is the ordinary time). The differential system (1) is then transformed into the '(x, s, K)-system'

\[ \ddot{x}' - \frac{1}{r^2} (x, x')x' + \frac{K^2}{r} x = \text{Pert}, \quad t' = r. \]

(2)

The parenthesis denotes the scalar product and the new 'Pert' is obtained of course by correct transformation of the perturbing term in Equation (1). Accents denote differentiation with respect to \( s \).

The KS-transformation (Stiefel and Scheifele, 1971) introduces the 4-dimensional state vector \( u \) with the components \( u_1, u_2, u_3, u_4 \) and related to the \( x \)-vector by the set of formulae

\[ x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2, \quad x_2 = 2(u_1u_2 - u_3u_4), \]
\[ x_3 = 2(u_1u_3 + u_2u_4). \]

(3)

The new equations of motion are

\[ u'' + \frac{h}{2} u = \text{Pert}, \quad t' = |u|^2 \]

(4)

\(-h\) is the total energy of the moving particle. These equations are referred to as the \((u, s, h)\)-equations. (The gravitational constant \( K^2 \) has disappeared and was replaced by \( h \)). The Equations (4) must be supplemented by a differential equation for the element \( h \).
By a ‘near-parabolic’ or ‘low energy’ orbit is understood a situation where \((hu)\) is of the order of magnitude of the physical perturbations, thus the \(h\)-term in (4) is conveniently incorporated into the perturbations and thus shifted to the right-hand side:

\[
u'' = -\frac{h}{2} u + \text{Pert.} \quad t' = |u|^2.
\] (5)

The right-hand side may be called ‘mathematical perturbation’.

We discuss, at first, the unperturbed case. It is characterized by \(h=0\) and vanishing physical perturbations. The orbit is consequently a Keplerian parabolic orbit in \(x\)-space. In the \(u\)-space we have the set of equations

\[
u'' = 0, \quad t' = |u|^2.
\] (6)

This is a uniform rectilinear motion with respect to the independent variable \(s\). The orbit in the \(u\)-space is a straight line. The time \(t\) is, as function of \(s\), a cubic polynomial. This follows from the time-Equation (6). Any automatic computer does integrate the \(u\)- and \(t\)-Equations (6) without truncation error and hopefully this good behaviour carries over to the perturbed case such that truncation errors remain small.

The four variables \(u_j\) are elements since they vary linearly in unperturbed motion. If we put

\[
u = \alpha + \beta s
\] (7)

the vectorial elements \(\alpha, \beta\) are constant during the unperturbed motion and they vary slowly during a perturbed motion, provided the perturbing forces are small.

This set of \(u\)-elements is very convenient for near-parabolic orbits. At first they obey the set (5) of differential equations of second order. This fact permits taking advantage of the special numerical techniques which were established for equations of second order. This is in contrast to the classical elements which satisfy a system of first order which cannot be reduced to half as many second order equations. Furthermore the transformations (3) from elements to rectangular coordinates are simple and straightforward and do not need trigonometric functions.

There was some controversy among the experts whether coordinates or elements should be used for numerical purposes. In our case such a discussion is almost meaningless, since the distinction between coordinates and elements disappears. The \(u_j\) are elements as well as coordinates in the four-dimensional space.

The \((u, s, h)\)-Equations (5) can be subjected to the transformation (3) thus producing \(x\)-differential equations. The resulting set is by no means the \((x, s, K)\)-set (2) since the former contains the varying energy \(h\) whereas the latter contains the constant \(K\). One obtains by the suggested transformation

\[
x'' = -\frac{1}{r^2} (x, x')x' + \frac{1}{2r^2} |x'|^2 x = -hx + \text{Pert.}
\] (8)