AN ESTIMATE FOR THE SOLUTIONS OF STOKES EQUATIONS IN EXTERIOR DOMAINS

P. Maremonti and V. A. Solonnikov

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A boundary value problem for the Stokes equations is examined in an exterior domain \( \Omega \subset \mathbb{R}^n \) with a uniform Dirichlet condition on the boundary and a homogeneous condition at infinity. It is shown that estimating the norm \( L^p(\Omega) \) of the second derivatives of the velocity vector field by the same norm of the exterior forces vector field is correct for \( p < n/2 \), but not for \( p \geq n/2 \). This estimate is valid also for \( p \geq n/2 \) if the boundary conditions are modified at infinity.

Let \( \Omega \subset \mathbb{R}^n \) be an unbounded external domain with a compact boundary \( \Gamma \subset \mathbb{C}^2 \) and with a simply connected complement \( G = \mathbb{R}^n \setminus \Omega \). We examine the problem

\[
\begin{align*}
- \Delta \vec{v}(x) + \nabla q(x) &= \vec{f}(x), & \vec{u} \cdot \nabla \vec{v}(x) &= 0, & x \in \Omega, \\
\vec{v} &= 0, & x \in \Gamma, \\
\vec{v} &\to 0, & |x| &\to \infty \quad (n > 2), \\
\vec{v} &\to \vec{v}_0 = \text{Const}, & |x| &\to \infty \quad (n = 2).
\end{align*}
\]

It is known (see [1] for example) that for a smooth finite \( f(x) \) this problem has a solution which vanishes at infinity (in the case \( n > 2 \)) in a power-law fashion: \( \vec{v}(x) = O(|x|^{2-n-|\alpha|}) \) for \( n = 2 \), \( \vec{v} = \vec{v}_0 + O(1/|x|) \) with a constant \( v_0 \). We are interested in the estimate

\[
\| \vec{v} \|_p + \| \nabla q \|_{p, \Omega} \leq C_1 \| \vec{f} \|_{p, \Omega},
\]

where \( \| \vec{f} \|_{p, \Omega} = \left( \int_{\Omega} |\vec{f}(x)|^p \, dx \right)^{1/p} \) is the norm in \( L^p(\Omega) \) and

\[
\| \vec{v} \|_p = \| D^2 \vec{v} \|_{p, \Omega} + \| D \vec{v} \|_{p, \Omega} + \| \vec{v} \|_{p, \Omega_1},
\]

and \( \Omega_1 \) is a bounded region in \( \mathbb{R}^n \) such that its complement \( \mathbb{R}^n \setminus \Omega_1 \) consists of two nonintersecting components: \( G \) and \( \mathbb{R}^n \setminus (G \cup \Omega_1) \) (we can assume \( \Omega_1 = \{ x \in \Omega : |x| < r \} \) with a suitable \( r > 0 \)).

We denote by \( W_p(\Omega) \) closed sets [in the norm (4)] of vector fields \( \vec{u} \in W_p^2(\Omega) \) with compact carriers. It is easy to show that \( W_p(\Omega) \) does not depend on the choice of the domain \( \Omega_1 \) and that vector fields belong to this subdomain. The vector fields which can be summed quadratically along with their first and second derivatives over any bounded subdomain \( \Omega \). At large \( |x| \) they behave the same as solutions to the problem (1) and (2) with a finite \( f(x) \). Moreover, for \( p \geq n/2 \), the space \( W_p(\Omega) \) belongs to a constant vector field \( \vec{u} = a \), but if \( p \geq n \) then \( \vec{u} = a + \sum_{i=1}^n b_i x_i \in W_p(\Omega) \), \( a = \text{const} \), \( b_i = \text{const} \). The corresponding approximating sequence will be \( \vec{u}(x) = \omega_m(x) \), where \( \omega_m(x) = \zeta(x/m) \), \( \zeta \in C_0^\infty(\mathbb{R}^n), \zeta(x) = 1 \) for \( |x| \leq 1 \) and \( \zeta(x) = 0 \) for \( |x| \geq 2 \); and in the limiting cases \( p = n/2 > 1 \) for \( \vec{u} = a \) and \( p = n \) for \( \vec{u} = \sum_{i=1}^n b_i x_i \omega_m(x) = \psi((\ln |x| -...
ln m)/(k - 1)ln m], where k > 1, and \( \psi(t) \) is an infinite differentiable function, which vanishes for \( t \geq 1 \) and is unity for \( t \leq 0 \). If \( p < n/2 \), then \( a = \overline{w}_p(\Omega) \) but for \( p < n \). \( \sum_j b_j x_j \overline{w}_p(\Omega) \) for nonzero \( a \) and \( b \). Otherwise, from the basic theory presented by Sobolev we would have \( \frac{1}{p_1} = \frac{1}{p} - \frac{2}{n} \), \( \frac{1}{p_2} = \frac{1}{p} - \frac{1}{n} \).

Let \( T(v, q) \) be a stress tensor (its elements are \( T_{ij} = -\delta_{ij} q + \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial x_j} \)) and let \( E_{ij}(x) \) and \( P_j(x) \) be elements of the fundamental matrix of the Stokes equations. They satisfy the relationship

\[
-\Delta E_{ij} + \frac{\partial P_j}{\partial x_i} = \delta_{ij} \delta(x), \sum_{i=1}^n \frac{\partial E_{ij}}{\partial x_i} = 0
\]

and are given by the formulas

\[
E_{ij}(x) = \frac{1}{2|S_1|} \left( \frac{1}{n-2} \left( \frac{S_{ij}}{|x|^{n-2}} + \frac{x_i x_j}{|x|^n} \right) \right), \quad P_j(x) = \frac{1}{|S_1|} \frac{x_j}{|x|^n} \quad (n > 2),
\]

\[
E_{ij}(x) = \frac{1}{4\pi} \left( \frac{S_{ij} \ln \frac{1}{|x|} + x_i x_j}{|x|^n} \right), \quad P_j(x) = \frac{1}{2\pi} \frac{x_j}{|x|^n} \quad (n = 2),
\]

in which \( |S_1| \) is the surface area of a unit sphere \( S_1 \subset \mathbb{R}^n \). We use \( E_j(x) \) to represent a vector with components \( \{E_{ij}\}_{i=1,2,3} \), and \( \mathcal{E}(x) \) to represent a matrix with elements \( E_{ij}(x) \).

Our results reduce to the following.

**THEOREM 1.** The estimate (3) is valid for \( 1 < p < n/2 \) and does into occur for \( p \geq n/2 \). It is also satisfied in the case of \( f \) has a compact carrier and satisfies the additional condition

\[
\int_{\Omega} \Phi(x_0-y) \overline{f}(y) dy = 0 \quad (\frac{1}{2} < p < n, \quad n > 2),
\]

\[
\int_{\Omega} \Phi(x_0-y) \overline{f}(y) dy = 0, \quad \int_{\Omega} \overline{f}(y) dy = 0 \quad (1 < p < n, \quad n = 2),
\]

but in the case \( p \geq n \), moreover

\[
\int_{\Omega} \Phi(x_0-y) \overline{f}(y) dy = 0 \quad (i = 1, \ldots, n)
\]

for any \( x_0 \in \Omega \).

From theorem 1 it follows that for all \( f \in L_p(\Omega), \quad 1 < p < n/2 \), problem (1) and (2) has a natural solution \( v \in w_p(\Omega), \quad \nabla q \in L_p(\Omega) \) and it satisfies the estimate (3). The condition (2) is valid because, due Sobolev's theorem, \( v \in L_r(\Omega) \) with \( 1/r = 1/p - 2/n \). For \( p \geq n/2 \), this condition loses sense. The estimate (3) turns out to be correct for solving another problem.

**THEOREM 2.** For any \( f \in L_p(\Omega) \) a \( v \in w_p(\Omega) \) and a \( \nabla q \in L_p(\Omega) \) exist which satisfy Eq. (1) and the complementary conditions (\( n \) is the outward normal to \( \Gamma \) relative to \( \Omega \)):

\[
\int_{\Gamma} \Phi(x-x_0) T(\overrightarrow{v}, q) \overline{w}(x) dS = 0 \quad \text{for} \quad \frac{1}{2} \leq p < n, \quad n > 2,
\]

\[
\int_{\Gamma} T(\overrightarrow{v}, q) \overline{w}(x) dS = 0 \quad \text{for} \quad 1 < p < n = 2,
\]

and in the case \( p \geq n \) and \( n \geq 2 \) also

\[
\int_{\Gamma} \frac{\partial \Phi(x-x_0)}{\partial x_i} T(\overrightarrow{v}, q) \overline{w}(x) dS = 0, \quad i = 1, \ldots, n.
\]

This solution to the problem (1) and (6) is unique, and satisfies the estimate (3).

Analogous results for the exterior Dirichlet problem for the equation \( -\Delta u = f \) were obtained in [2].

First we take the case \( n > 2 \). We show two auxiliary propositions. In the first we examine linear algebraic systems

\[
\sum_{i=1}^n \int_{\Delta_i} E_{ij} \omega \overline{\omega} \overline{\xi}_i = -\xi_i
\]