AN INEQUALITY FOR ACUTE TRIANGLES

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An inequality relating the tangents of half angles of a triangle that is necessary for solving Malfatti's problem is proved.

In [1] Los' proposed a method for solving the old problem of how to arrange three non-overlapping circles of maximum area in a triangle. However, he did not succeed in actually solving this problem in [1]. The solution is given in Zalgaller and Los' s paper found in the present issue of the Ukrainskii Geometricheskii Sbornik. One of its more laborious steps was overcome using an inequality proposed in the present notice, an inequality that is also interesting by itself.

One of the more time-consuming steps is proving that the so-called Malfatti circles have a total area less than that of an inscribed circle and its two complementary circles which are enclosed in the smaller angles of a triangle. The goal of this notice is to establish an inequality, which is interesting by itself, and which also enables us to drastically simplify this step in [1].

**THEOREM.** The following holds for an acute triangle with angles $2\alpha, 2\beta, 2\gamma$:

$$
\frac{(1+\tan \alpha)^4 + (1+\tan \beta)^4 + (1+\tan \gamma)^4}{(1+\tan \alpha)^2(1+\tan \beta)^2(1+\tan \gamma)^2} \leq \frac{2}{3} \left( \tan^4 \alpha + \tan^4 \beta + \tan^4 \gamma \right),
$$

where

$$
C = \frac{9}{(\sqrt{3}+1)^2} - \frac{2}{9} = 0.98355.
$$

Equality in (1) is attained only for an equilateral triangle.

We preface the proof with two lemmas.

**LEMMA 1.** On the interval $0, 4 \leq x \leq 1$, the smooth function

$$
f(x) = \frac{4}{3} x^2 \left( 1-x^2 \right)^4 \left( \frac{\left( 1+2x-x^2 \right)}{24x^4} \right) - \frac{8x^2}{\left( 1+2x-x^2 \right)^2} - \frac{(1+2x-x^2)^2}{4x^2(1+x)^4},
$$

strictly decreases on $0.4 \leq x \leq 1/\sqrt{3}$ and strictly increases on $1/\sqrt{3} \leq X \leq 1$. Therefore, on $0.4 \leq x \leq 1$ $f$ attains an absolute minimum at $x = 1/\sqrt{3}$. This minimum is equal to

$$
f \left( \frac{1}{\sqrt{3}} \right) = \frac{2}{9} - \frac{9}{(\sqrt{3}+1)^2} = -C.
$$

**Proof.** Let us denote by $A(x)$ the sum of the first two terms on the right-hand side of (3) and by $B(x)$ the sum of the remaining two terms. Straightforward calculations lead us to the following derivatives $A'(x)$ and $B'(x)$:

$$
A'(x) = \frac{(3x^2-1)(11x^4+3x^2+1)}{6x^5},
$$

$$
B'(x) = \frac{(3x^2-1)(11x^4+49x^2+140x^2+148x^2+194x^2+214x^4+140x^2+52x^2+11x^4)}{2x^2(1+x)^5(1+2x-x^2)^3}.
$$

whence

\[ f'(x) = A'(x) + B'(x) = (3x^2 - 1) \frac{\varphi(x)}{\psi(x)}, \]

where

\[ \varphi(x) = -11x^{17} + 11x^{16} + 118x^{15} + 14x^{14} - 474x^{13} - 546x^{12} + 301x^{11} + 1091x^{10} + 892x^9 + 424x^8 - 132x^7 - 408x^6 - 234x^5 + 22x^4 + 89x^3 + 47x^2 + 11x + 1; \]

\[ \psi(x) = 6x^5(1+x)^3(1+2x-x^2)^2. \]

The validity of Lemma 1 follows from (5) and the fact that for \(0.4 \leq x \leq 1\) we certainly have \(\varphi(x) > 0\) and \(\psi(x) > 0\). The latter of these two inequalities is obvious and the former can be verified, for example, by writing \(\varphi(x)\) in the form of a sum of terms that are positive and negative on \(0.4 \leq x \leq 1\): \(\varphi(x) = (1-x^3)(22x^4 + 68x^3 + 39x^2) + (1-x^5)^2 + 11x(1-x^7)^2 + 8x^2(1-x^6)^2 + 21x^3(1-x^5)^2 + 220x^4(x-0.4) + x^6(408x^2 - 65) + x^8(441x^2 - 70) + x^8(141x^3 - 9) + x^9(132 - 39x - 68x^2 - 22x^3) + 3x^{17} + x^{11}(940 + 814x - 55x^2 - 579x^3 - 913x^4 - 167x^5).\)

**Lemma 2.** The theorem is true in the class of acute isosceles triangles. Indeed let \(2\alpha, 2\beta, 2(\pi/2 - 2\alpha)\) be the angles of an acute isosceles triangle. Here \(\pi/8 < \alpha < \pi/4\). Let us denote \(t = x, \sqrt{2} - 1 < x < 1\). Then for our triangle (1) takes the form

\[
\frac{8x^2}{(1+2x-x^2)^2} + \frac{(1+2x-x^2)^2}{4x^2(1+x)^4} = C \left[ \frac{2}{3} \left\{ \frac{1}{2} \left( 1-x^2 \right) \right\} \right],
\]

and the validity of Lemma 2 follows from Lemma 1.

**Proof of the Theorem.** Let \(p, R, r\) be, respectively, the half-perimeter, the radius of the circumscribed and the radius of the inscribed circles of the triangle under consideration. As is known [2, p. 30], \(\tan \alpha, \tan \beta, \tan \gamma\) are the three roots of \(py^3 - (4R + r)y^2 + py - r = 0\). Owing to similarity, we assume that

\[ 4R + r = 1. \]

Then the basic symmetric functions are

\[ \sigma_1 = \tan \alpha + \tan \beta + \tan \gamma = \frac{1}{p}, \quad \sigma_2 = \tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1, \quad \sigma_3 = \tan \alpha \tan \beta \tan \gamma = \frac{r}{p}. \]

It enables us to express any symmetric polynomial in \(\tan \alpha, \tan \beta, \tan \gamma\) in terms of \(r\) and \(p\). In particular,

\[ \tan^4 \alpha + \tan^4 \beta + \tan^4 \gamma = \frac{1}{p^4} \left( 4r^2 + 2p^4 - 4p^2 + 1 \right), \quad (1+\tan \alpha)^4 + (1+\tan \beta)^4 + (1+\tan \gamma)^4 = \frac{1}{p^4} \left( 4r^2 (3p^2 + 1) - 7p^4 - 8p^3 + 2p^2 + 4p + 1 \right), \quad (1+\tan \alpha) (1+\tan \beta) (1+\tan \gamma) = \frac{1}{p} (r + 2p + 1). \]

Let us trace the range of \(r\) and \(p\) for acute triangles satisfying (6). From \(R \geq 2r\) (see [2, p. 8]) and (6) we have \(0 < r \leq 1/9\). For a fixed \(r\) and \(R = (1-r)/4\) the value of \(p\) (see [2, pp. 12-13]) can vary from

\[ p_{\text{min}}(r) = \frac{\sqrt{9r + 8 - 9r}}{8} \]

to

\[ p_{\text{max}}(r) = \frac{\sqrt{9r + 8 - 9r}}{8} \]

here \(p_{\text{max}}\) is attained for acute isosceles triangles. Let us note in passing that \(p < \pi R = \pi(1-r)/4 < \pi/4 < 1\).

The acuity of the triangle (see [2, p. 38]) implies that \(p > 2R + r = (1 + r)/2\). Thus, the range of \(i, p\) is the domain \(\Omega\) depicted in Fig. 1.

Let us now consider in \(\Omega\) the function