TOTALLY GEODESIC FOLIATIONS CLOSE TO RIEMANNIAN FOLIATIONS

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The relation of the curvature and topology of totally geodesic foliations close to Riemannian ones is studied. The main result complements Ferus's famous theorem on totally geodesic foliations.

In this paper we consider an \((n+v)\)-dimensional Riemannian manifold \(M^{n+v}(v, n, 0)\) with a \(v\)-dimensional foliation \(\{L^x\}\) (see [1]). In [2] the theorem was proved that if the leaves are complete and totally geodesic and the mixed sectional curvature
\[
K(x, y) = \text{const} > 0 \quad (x \in TL, y \in TL^x),
\]
then
\[
v < \rho(n),
\]
where \(\rho(n) - 1\) is the maximum number of continuous, pointwise linear, independent vector fields on the sphere \(S^n\).

In [3] it was noted that (2) is exact but is violated when (1) is replaced by the weaker condition
\[
K(x, y) > 0 \quad (x \in TL, y \in TL^x),
\]
In the present paper we prove the validity of (2) for a totally geodesic foliation with condition (3) and being "close" in a certain sense to a Riemannian or a conformal foliation.

Important information about \(\{L\}\) is contained in the second fundamental form \(h^\perp: TL^\perp \times TL^\perp \rightarrow TL\) of the orthogonal distribution \(TL^\perp\) and the corresponding symmetric operator \(W^\perp(x): TL^\perp \rightarrow TL^\perp(x \in TL)\) specified by (see [1]):
\[
h^\perp(y, z) = \frac{1}{2} (\nabla_y z + \nabla_z y)^T, \quad <W^\perp(x)y, z > = <h^\perp(y, z), x>.
\]
For a Riemannian foliation, \(h^\perp\) and \(W^\perp(x)\) are zero and for a conformal one \(W^\perp(x)\) is a homothetic transformation.

Let us denote by \(k_{\min}(p)\) the minimum of the foliation's mixed sectional curvature at the point \(p \in M\).

THEOREM. Let \(M^{n+v}(v, n > 0)\) be a Riemannian manifold with foliation \(\{L^x\}\) into complete, totally geodesic leaves and suppose that at every \(p \in M\)
\[
|W^\perp(x)y|^2 \leq k_{\min}(p) \quad (x \in TL^\perp, y \in TL^\perp, |x| = |y| = 1).
\]
is fulfilled. Then, if a) \(v \geq \rho(n)\) and b) \(M, \{L\}\) are Kählerian manifolds and \(v > 2\) for \(n\) divisible by 4, then \(k_{\min} = 0\) and \(\{L\}\) is a Riemannian foliation.

COROLLARY. Let \(M^{n+v}(v, n > 0)\) be a Riemannian manifold with a totally geodesic foliation \(\{L^x\}\) and suppose that on a complete leaf \(L_0\)
\[
|W^\perp(x)y|^2 \leq K(x, y) \quad (x \in TL_0, y \in TL^x_0, |x| = |y| = 1).
\]
Then $\nu < \rho(n)$, and if $M$, $\{L\}$ are Kählerian manifolds, then $\nu = 2$ and $n$ is divisible by 4.

Remarks. 1. The estimate of the leaf's dimension in the Kählerian case of the Theorem and the Corollary is exact because a Riemannian submersion with 2-dimensional totally geodesic fibers $CP^1 \subset CP^2$ exists (see [4]). 2.

Condition (5) can be relaxed with the following requirement: there exists a $p \in L_0$ such that along any naturally parametrized geodesic $\gamma : R \to L_0$, $\gamma(0) = p$ we have $|W^\perp(\gamma(t))y|^2 \leq K(\frac{\gamma(t)}{t}, y)$, $y \in T_{\gamma(t)}L_0^\perp$, $|y| = 1$, $t \in R$. 3. The Theorem and the Corollary (as well as Theorems 3 and 4 of [3]) admit a natural strengthening for foliations close to conformal ones.

THEOREM'. Let $M^{n+p}$ ($n > 0$) be a Riemannian manifold with foliation $\{L^p\}$ into complete totally geodesic leaves and suppose that there exists a smooth linear functional $\beta : TL \to R$ for which

$$|W^\perp(x)y - \beta(x)y|^2 \leq k_{\text{min}}(p)$$

Then if a) $\nu \geq \rho(n)$ ($\nu = 1$, $n$ is odd) and b) $M$, $\{L\}$ are Kählerian manifolds and $\nu > 2$ when $n$ is divisible by 4 ($\nu = 2$, $n$ is not divisible by 4), then $k_{\text{min}} = 0$, and the foliation is conformal (Riemannian).

COROLLARY'. Let $M^{n+p}$ ($n > 0$) be a Riemannian manifold with a totally geodesic foliation $\{L^p\}$ and suppose that for a complete leaf $L_0$ there exists a smooth linear functional $\beta : TL \to R$ with

$$|W^\perp(x)y - \beta(x)y|^2 \leq K(x,y)$$

Then $\nu < \rho(n)$ and if $M$, $\{L\}$ are Kählerian manifolds, then $\nu = 2$ and $n$ is divisible by 4.

Proof of the Theorem and the Corollary. In [2] and [5], for a totally geodesic foliation $\{L\}$ on $M$ the bilinear operator $B : TL \times TL^\perp \to TL^\perp$ was determined according to

$$B(x, y) = (\nabla_x \bar{x})^\perp (xTL, yTL^\perp),$$

where $\bar{x} \subset TL$ is a smooth local vector field containing $x$ and

$$(\nabla_x B)(x, y) + B(x, B(x, y)) + R(y, x)x = 0. \tag{7}$$

holds. Let us denote by $B^+(x, \cdot)$ and $B^-(x, \cdot)$ the parts of (6) that are symmetric and cosymmetric in $y$ and note that $B^+(x, \cdot) = -W^\perp(x)$. Isolating in (7) parts symmetric and cosymmetric in $y$, we obtain

$$(\nabla_x B^+)(x, y) + B^+(x, B^+(x, y)) + B^-(x, B^-(x, y)) + R(y, x)x = 0.$$ \tag{8}

If $\nu \geq \rho(n)$ ($\nu > 2$ for $n$ divisible by 4 in the Kählerian case), then for every $p \in L$ there exist unit vectors $x \in T_pL$, $y \in T_pL^\perp$ and a number $\lambda$ with $B^-(x, y) = \lambda y$. Moreover, thanks to the cosymmetry of $B^-(x, \cdot)$ we have $\lambda = 0$ and, in particular, $\det B^-(x, \cdot) = 0$. Indeed, if $B^-(x, \cdot)$ does not have an eigenvector for any nonzero $x \in T_pL$, then we choose an orthonormal basis $\{e_i\} \subset T_pL$ and, following [2], construct $v$ linearly independent vector fields $\{W_i\}$, tangent to $S^{n-1} \subset T_pL$ according to the rule $w_i(y) = B^-(e_i, y)$. In the Kählerian case the existence of an eigenvector of $B^-$ is guaranteed by the following lemma.

**Lemma [6].** Let $I : R^n \to R^n$, $I : R^n \to R^n$ be complex structures on Euclidean spaces and let $D : R^n \times R^n \to R^n$ be a bilinear operator with the property $D(IX, y) = ID(x, y)$. Then for $\nu > 2$ there exist unit vectors $x \in R^n$, $y \in R^n$ and a number $\lambda$ such that $D(x, y) = \lambda y$.

Note that for a Riemannian foliation (8a) takes on the simple form $B^-(x, B^-(x, y)) + R(y, x)x = 0$ and for the $x$, $y$ we immediately obtain $K(x, y) = 0$, that is, $k_{\text{min}}(p) = 0$.

Let us consider (8b) along a naturally parametrized geodesic $\gamma : R \to L$ ($\gamma(0) = p$, $\gamma(0) = x$),

$$B_{\epsilon_0}^+ + B_{\epsilon_0}^- = 0.$$ \tag{9}

where $B_{\epsilon_0}^+ = B^+(\gamma(t), \cdot)$, $B_{\epsilon_0}^- = B^-(\gamma(t), \cdot)$. 

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