NUMERICAL METHODS, INVESTIGATION AND SOLUTION OF EQUATIONS

BOUNDARY-VALUE PROBLEMS FOR x-ANALYTICAL FUNCTIONS WITH WEIGHTED BOUNDARY CONDITIONS

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We consider boundary-value problems for x-analytical functions of a complex variable \( z = x + iy \) in a number of domains. Limit values with the weight \((\ln x)^{-1}\) are given for the real part of the x-analytical function on the sections of the boundary that follow the imaginary axis, and simple limits are given for the real part of the x-analytical functions on the part of the boundary outside the imaginary axis. The apparatus of integral representations of x-analytical functions is applied to obtain a solution of the problem in quadratures.

The system of differential equations for the real and imaginary parts of an x-analytical function of a complex variable \( z = x + iy \) degenerates on the sections of the boundary that follow the imaginary axis. In boundary-value problems for x-analytical functions, it is impossible to specify the limit values of the real part of the x-analytical function on these sections of the boundary, because the boundary-value problem is ill-posed in this case. The limit values of the real part of an x-analytical function therefore should be specified with the weight \((\ln x)^{-1}\) on sections of the boundary that follow the imaginary axis. In this paper, we apply the apparatus of integral representation of x-analytical functions by analytical functions [1, 2] and the ensuing procedure for solving boundary-value problems [2-6] in order to derive a solution in quadratures of a number of problems for x-analytical functions with weighted boundary conditions. Boundary-value problems with weighted boundary conditions for degenerating differential equations, and in particular for the Euler–Poisson–Darboux equation, have been previously considered in [7-11].

1. Let \( G_I = \{z = x + iy; x > 0\} \); \( a_k, b_k \) (\( k = 1, \ldots, n \)) are points of the imaginary axis, \( a_k < b_k < a_{k-1} < b_{k-1} \) (\( k = 2, \ldots, n \)).

Problem 1. Find a function \( \tilde{\Phi}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y) \), which is x-analytical in \( G_I \) and satisfies the boundary conditions

\[
\lim_{y \to 0} (\ln y)^{\frac{1}{k}} \tilde{u}(x, y) = \Phi_k(y), \quad a_k \leq y \leq b_k \quad (k = 1, \ldots, n) \tag{1}
\]

\[
\tilde{v}(0, y) = \begin{cases} 0, & 0 < y < \infty, \\ \Phi_k(b_k), & b_k < y < a_{k-1} = -\infty, \quad (k = 1, \ldots, n). \end{cases} \tag{2}
\]

where \( \Phi_k(y) \) (\( k = 1, \ldots, n \)) are given functions satisfying the Hölder condition and such that

\[
\Phi_k(a_k) = \Phi_k(b_k) = 0 \quad (k = 1, \ldots, n) \tag{3}
\]

\( D_k \) (\( k = 1, \ldots, n \)) are real constants (not specified in advance).

The solution of the various problems in this paper is sought on the set \( M_I \) of functions \( \tilde{\Phi}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y) \) with the following properties: 1) \( \tilde{\Phi}(z) \) is continuous on the part of the boundary outside the imaginary axis; 2) \( \tilde{v}(0, y) \) and \( \lim_{y \to 0} (\ln y)^{-1} \tilde{u}(x, y) \) are continuous on the sections of the imaginary axis included in the boundary; 3) in an infinite domain, \( \tilde{\Phi}(z) \) is bounded at infinity and its real part tends to zero as \( |z| \to \infty \).

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The solution of problem 1 using the basic integral representation of x-analytical functions \([2]\) is sought in the form

\[
\mathcal{F}(z) = \text{Re} \int_{z_0}^{z} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1/2}(z+\zeta)^{1/2}} + i \text{Im} \int_{z_0}^{z} \frac{f(\zeta)(\zeta-iy)d\zeta}{(z-\zeta)^{1/2}(z+\zeta)^{1/2}}.
\]  

(4)

Here the function \(f(z) = u(x, y) + iv(x, y)\) is analytical in \(G_1\), continuous on the boundary, satisfies the condition

\[
v(0, y) = 0, \quad b_{k+1} < y < a_k \quad (k = 0, \ldots, n; \quad a_0 = \infty, \quad b_{n+1} = -\infty)
\]

(5)

and has a zero of at least first order at infinity. For a fixed \(z\), we denote by \((z - \zeta)^{1/2}(z + \zeta)^{1/2}\) a function which is single-valued in the plane with a cut from \(-\bar{z}\) to \(z\) intersecting the imaginary axis in the interval \(b_i < y < \infty\) and has \(\arg(z - \zeta)(z + \zeta)^{1/2} = 0\) at the points of the imaginary axis above the cut (this branch of the function \((z - \zeta)^{1/2}(z + \zeta)^{1/2}\) is denoted \(R(z, \bar{z}, \zeta)\)). Integration in (4) is over a contour in \(G_1\) that joins an arbitrary (non-fixed) point \(z_0 = iy_0\) on the imaginary axis above the cut with the point \(z\) without intersecting the cut that defines the single-valued branch \(R(z, \bar{z}, \zeta)\).

Following \([3]\), we express the function \(\tilde{f}(z)\) defined by the integral representation (4) in terms of the limit values of the analytical function \(f(z)\). We have

\[
\tilde{f}(z) = \left\{ \text{Re} \int_{x = 0}^{x = \infty} \frac{f^- (\zeta)d\zeta}{R(z, \bar{z}, \zeta)} + i \text{Im} \int_{x = 0}^{x = \infty} \frac{f^- (\zeta)(\zeta-iy)d\zeta}{R(z, \bar{z}, \zeta)} + iD \right\}.
\]

(6)

where \(f^- (\zeta)\) is the limit of the function \(f(z)\) as we approach the imaginary axis, \(D\) is a real constant such that

\[
D = D_n = -\pi \text{Im} \text{Res} f(z) ; \quad \lim_{z \to \infty} \arg R(z, \bar{z}, iak) = -\pi \quad (k = 1, \ldots, n).
\]

(7)

Formula (6) can be represented in the form

\[
\mathcal{F}(z) = \frac{1}{2} \left\{ \sum_{k=1}^{n} \int_{0}^{b_k} \frac{v(\eta, y)}{[x^2 + (\eta-y)^2]^{1/2}} d\eta + iD \right\}.
\]

(8)

Let us investigate the behavior of the real part of the function \(\tilde{f}(z)\) defined by equality (8) as \(x \to 0\), \(a_j \leq y \leq b_j\). To this end we assume that \(v(0, y)\) satisfies the Hölder condition on the interval \([a_j, b_j]\) and represent \(\tilde{u}(x, y)\) in the neighborhood of the interval \([a_j, b_j]\) in the form

\[
\tilde{u}(x, y) = \frac{1}{2} \left\{ \int_{0}^{b_j} \frac{v(\eta, y) - v(0, y)}{[x^2 + (\eta-y)^2]^{1/2}} d\eta - 2v(0, y)\ln x + w(0, y) \ln \left\{ \frac{(b_j - y) + \sqrt{x^2 + (b_j - y)^2}}{2x} \right\} - \frac{(a_j - y) + \sqrt{x^2 + (a_j - y)^2}}{2x} \right\} + \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{v(\eta, y)}{[x^2 + (\eta-y)^2]^{1/2}} d\eta, \quad a_j \leq y \leq b_j.
\]

(9)

From (9) it follows that as \(x \to 0\), \(y \in [a_j, b_j]\) the function \(\tilde{u}(x, y)\) in general has a logarithmic singularity, and

\[
\lim_{x \to 0} (\ln x)^{-1} \tilde{u}(x, y) = \begin{cases} \frac{-1}{2}v(0, a_j), & y = a_j, \\ \frac{-1}{2}v(0, y), & a_j < y < b_j, \\ \frac{-1}{2}v(0, b_j), & y = b_j. \end{cases}
\]

(10)

The limit of the function \(\tilde{u}(x, y)\) as \(x \to 0\), \(y \in [a_j, b_j]\) exists if \(v(0, y) = 0\), \(y \in [a_j, b_j]\).

From equalities (10), using the boundary conditions (1), we obtain

\[
v(0, y) = -\Phi_j(y), \quad y \in [a_j, b_j] \quad (j = 1, \ldots, n).
\]

(11)

Substituting the values \(v(0, y)\) in the integral representation (8), we obtain the solution of problem 1 in the form

\[
\mathcal{F}(z) = \frac{1}{2} \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{\Phi_k(y) [1 + i(\eta - y)]}{[x^2 + (\eta-y)^2]^{1/2}} d\eta + iD.
\]