NUMERICAL METHODS, INVESTIGATION AND SOLUTION OF EQUATIONS

BOUNDARY-VALUE PROBLEMS FOR x-ANALYTICAL FUNCTIONS WITH WEIGHTED BOUNDARY CONDITIONS

A. A. Kapshivyi

We consider boundary-value problems for x-analytical functions of a complex variable $z = x + iy$ in a number of domains. Limit values with the weight $(\ln x)^{-1}$ are given for the real part of the x-analytical function on the sections of the boundary that follow the imaginary axis, and simple limits are given for the real part of the x-analytical functions on the part of the boundary outside the imaginary axis. The apparatus of integral representations of x-analytical functions is applied to obtain a solution of the problem in quadratures.

The system of differential equations for the real and imaginary parts of an x-analytical function of a complex variable $z = x + iy$ degenerates on the sections of the boundary that follow the imaginary axis. In boundary-value problems for x-analytical functions, it is impossible to specify the limit values of the real part of the x-analytical function on these sections of the boundary, because the boundary-value problem is ill-posed in this case. The limit values of the real part of an x-analytical function therefore should be specified with the weight $(\ln x)^{-1}$ on sections of the boundary that follow the imaginary axis. In this paper, we apply the apparatus of integral representation of x-analytical functions by analytical functions [1, 2] and the ensuing procedure for solving boundary-value problems [2-6] in order to derive a solution in quadratures of a number of problems for x-analytical functions with weighted boundary conditions. Boundary-value problems with weighted boundary conditions for degenerating differential equations, and in particular for the Euler–Poisson–Darboux equation, have been previously considered in [7-11].

1. Let $G_1 = \{z = x + iy; x > 0\}; i a_k, i b_k (k = 1, \ldots, n)$ are points of the imaginary axis, $a_k < b_k < a_{k-1} < b_{k-1}$ ($k = 2, \ldots, n$).

Problem 1. Find a function $\tilde{\eta}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y)$, which is x-analytical in $G_1$ and satisfies the boundary conditions

\[ \lim_{x \to 0} (\ln x)^{1/3} \tilde{u}(x, y) = \Phi_k(y), \quad a_k \leq y \leq b_k \quad (k = 1, n) \]

\[ \tilde{v}(0, y) = \begin{cases} \nu_0, & b_1 < y < \infty, \\ D_k, & b_k+1 < y < a_k \quad (k = 1, n; b_{k+1} = -\infty), \end{cases} \]

where $\Phi_k(y) (k = 1, \ldots, n)$ are given functions satisfying the Hoelder condition and such that

\[ \Phi_k(a_k) = \Phi_k(b_k) = 0 \quad (k = 1, n); \]

$D_k (k = 1, \ldots, n)$ are real constants (not specified in advance).

The solution of the various problems in this paper is sought on the set $M_1$ of functions $\tilde{\eta}(z)$ that are x-analytical in $G_1$ and satisfy the following properties: 1) $\tilde{\eta}(z)$ is continuous on the part of the boundary outside the imaginary axis; 2) $\tilde{v}(0, y)$ and $\lim_{x \to 0} (\ln x)^{-1}\tilde{u}(x, y)$ are continuous on the sections of the imaginary axis included in the boundary; 3) in an infinite domain, $\tilde{\eta}(z)$ is bounded at infinity and its real part tends to zero as $|z| \to \infty$.

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The solution of problem 1 using the basic integral representation of x-analytical functions [2] is sought in the form

\[ \mathcal{F}(z) = \text{Re} \int_{\gamma_0} \frac{f(\zeta) d\zeta}{(z-\zeta)^{1/2} (\zeta+\xi)^{1/2}} + i \text{Im} \int_{\gamma_0} \frac{f(\zeta) (\zeta-iy) d\zeta}{(z-\zeta)^{1/2} (\zeta+\xi)^{1/2}}. \]  

(4)

Here the function \( f(z) = u(x, y) + iv(x, y) \) is analytical in \( G_1 \), continuous on the boundary, satisfies the condition

\[ \nu(0, y) = 0, \quad b_{k+1} < y < a_k \quad (k = 0, \ldots, n; \quad a_0 = \infty, \quad b_{n+1} = -\infty) \]  

(5)

and has a zero of at least first order at infinity. For a fixed \( z \), we denote by \( (z - \zeta)^{1/2} (\zeta + \xi)^{1/2} \) a function which is single-valued in the plane with a cut from \( -\zeta \) to \( z \) intersecting the imaginary axis in the interval \( b_1 < y < c_1 \) and has \( \arg(z - \zeta)(\zeta + \xi) = 0 \) at the points of the imaginary axis above the cut (this branch of the function \( (z - \zeta)^{1/2} (\zeta + \xi)^{1/2} \) is denoted \( R(z, \zeta, \xi) \)). Integration in (4) is over a contour in \( G_1 \) that joins an arbitrary (non-fixed) point \( z_0 = iy_0 \) on the imaginary axis above the cut with the point \( z \) without intersecting the cut that defines the single-valued branch \( R(z, \zeta, \xi) \).

Following [3], we express the function \( \tilde{f}(z) \) defined by the integral representation (4) in terms of the limit values of the analytical function \( f(z) \). We have

\[ \tilde{f}(z) = \frac{1}{2} \left[ \text{Re} \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{f(\zeta)(\zeta - iy) d\zeta}{R(\zeta, z, \xi)} + i \text{Im} \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{f(\zeta)(\zeta - iy) d\zeta}{R(\zeta, z, \xi)} + iD \right]. \]  

(6)

where \( f(\zeta) \) is the limit of the function \( f(z) \) as we approach the imaginary axis, \( D \) is a real constant such that

\[ D = D_n = -\pi \text{Im} \text{Res} f(z); \quad \arg R(z, \zeta, i\alpha_k) = -\pi \quad (k = 1, \ldots, n). \]  

(7)

Formula (6) can be represented in the form

\[ \mathcal{F}(z) = \frac{1}{2} \left\{ \sum_{k=1}^{n} \int_{a_k}^{b_k} \nu(0, y) \frac{[1 + i(y - y)]}{[x^2 + (y - y)^2]^{1/2}} d\eta + iD \right\}. \]  

(8)

Let us investigate the behavior of the real part of the function \( \tilde{f}(z) \) defined by equality (8) as \( x \to 0, \quad a_j \leq y \leq b_j \). To this end we assume that \( v(0, y) \) satisfies the Hölder condition on the interval \( [a_j, b_j] \) and represent \( \tilde{u}(x, y) \) in the neighborhood of the interval \( [a_j, b_j] \) in the form

\[ \tilde{u}(x, y) = \frac{1}{2} \int_{a_j}^{b_j} \frac{[\nu(x, y) - \nu(0, y)]}{[x^2 + (y - y)^2]^{1/2}} d\eta - 2v(0, y) \ln x + \]  

\[ + v(0, y) \ln \left\{ \frac{[b_j - y + \sqrt{x^2 + (b_j - y)^2}]}{[y - a_j + \sqrt{x^2 + (a_j - y)^2}]} \right\} + \]  

\[ + \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{v(0, y) d\eta}{[x^2 + (y - y)^2]^{1/2}}, \quad a_j \leq y \leq b_j. \]  

(9)

From (9) it follows that as \( x \to 0, \quad y \in [a_j, b_j] \) the function \( \tilde{u}(x, y) \) in general has a logarithmic singularity, and

\[ \lim_{x \to 0} (\ln x)^{-1} \tilde{u}(x, y) = \begin{cases} \frac{-2v(0, a_j)}{a_j}, & y = a_j, \\ \frac{-v(0, y)}{a_j}, & a_j < y < b_j, \\ \frac{-v(0, b_j)}{b_j}, & y = b_j. \end{cases} \]  

(10)

The limit of the function \( \tilde{u}(x, y) \) as \( x \to 0, \quad y \in [a_j, b_j] \) exists if \( v(0, y) = 0, \quad y \in [a_j, b_j] \). From equalities (10), using the boundary conditions (1), we obtain

\[ v(0, y) = -\Phi_j(y), \quad y \in [a_j, b_j] \quad (j = 1, \ldots, n). \]  

(11)

Substituting the values \( v(0, y) \) in the integral representation (8), we obtain the solution of problem 1 in the form

\[ \mathcal{F}(z) = \frac{1}{2} \sum_{k=1}^{n} \int_{a_k}^{b_k} \Phi_k(y) \frac{[1 + i(y - y)]}{[x^2 + (y - y)^2]^{1/2}} d\eta + iD. \]