DIFFUSION APPROXIMATION OF SYSTEMS WITH REPEATED CALLS AND AN UNRELIABLE SERVER

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We consider single-server queueing systems with repeated calls and an unreliable server, which may fail both when free and when busy. A central limit theorem and a diffusion approximation theorem are obtained for the queue as a time-dependent process in the case of a low rate of repeated calls.

Models of queueing systems with repeated calls arise in the study of computer networks and information transmission systems. Various models of multiple-access local-area networks have been considered in [1-3]. In this paper, we examine the asymptotic properties of transient modes in such systems assuming an unreliable server under a high load.

Consider a single-server system whose characteristics depend on \( n \) \( \rightarrow \infty \) and which receives a stream of incoming calls on its input. If the server is free and is in a normally operating state, the call begins processing. If the server is busy or faulty, the call is enqueued forming a source of repeated calls. Denote by \( Q_n(t) \) the number of sources of repeated calls at time \( t \). Let \( t_{n1} < t_{n2} < \ldots \) be the successive instants when processing is completed or when the faulty server is restored, \( t_{n0} = 0 \) and \( Q_{nk} = Q_n(t_{nk} + 0) \). Assume that the functions \( \lambda(a) \), \( \nu(a) \), and \( \mu(a) \), \( a \geq 0 \), are given such that on the interval \( [t_{nk}, t_{nk+1}) \) the stream of primary calls is Poisson with intensity \( \lambda_n(Q_{nk}) = \lambda(1/nQ_{nk}) \), and the arrival rate at the server of each enqueued call independently of the other calls is \( \mu_n(Q_{nk}) = 1/n(1/nQ_{nk}) \).

We assume that in the free state the server may fail at a rate \( \mu_n(Q_{nk}) = \mu(1/nQ_{nk}) \). The server also may be in a busy state. Denote by \( \theta_n \) the time between failures when the server is processing a call and by \( \kappa_n \) the service time of primary or repeated calls. Let \( Z_n = \theta_n \wedge \kappa_n \). If \( \kappa_n \leq \theta_n \), then the call is processed and leaves the system. If \( \theta_n < \kappa_n \), then the call is preempted and the preempted call forms a source of repeated calls with probability \( 0 \leq \alpha \leq 1 \) and leaves the system with probability \( 1 - \alpha \). When the server fails, it immediately begins to be repaired. Denote by \( V_n \) and \( U_n \) the repair times of a server that failed in the free and the busy state, respectively.

In what follows, the distribution function of the random variable \( \xi \) is denoted by \( F_\xi(x) \), and \( M_\xi = m_\xi \), \( D_\xi = \sigma_\xi^2 \).

**THEOREM 1.** If the functions \( \lambda(a) \), \( \nu(a) \), and \( \mu(a) \) are locally Lipschitz, \( \lambda(a) > 0 \), \( \nu(a) > 0 \), \( \mu(a) > 0 \), \( a > 0 \), \( \lambda(a) + \nu(a) + \mu(a) \leq L(1 + a) \), the random variables \( \theta_n \), \( \kappa_n \), \( U_n \), \( V_n \) are uniformly integrable, and for \( n \rightarrow \infty \)

\[
m_{12} \rightarrow m_{12}, \ m_{2n} \rightarrow m_{2n}, \ m_{n1} \rightarrow m_{n1}, \ m_{n2} \rightarrow m_{n2}, \ m_{2(\theta_n < \kappa_n)} \rightarrow \beta, \ \frac{1}{n}Q_n(0) \rightarrow \eta_0, 
\]

then for every \( T > 0 \)

\[
\sup_{t \in [0, T]} \left| \frac{1}{n}Q_n(nt) - \eta_0(t) \right| \rightarrow 0, 
\]

where \( \eta(t) \) satisfies the equation

\[
\frac{d\eta(t)}{dt} = \left[ \lambda(\eta(t)) - \mu(\eta(t)) \left(1 - \alpha\beta m(\eta(t))^{-1}\right) \right] dt,
\]

and \( W(t) \) is the standard Wiener process.

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The proof of the theorem relies on the results of [4, 5]. By construction of $Q_{nk}$ and $t_{nk}$, we may write $\theta_{nk+1} = \theta_{nk} + \xi_{nk}(Q_{nk}), t_{nk+1} = t_{nk} + \tau_{nk}, \tau_{nk}(Q_{nk})$, where the distributions of the random variables $\xi_{nk}(na)$ and $\tau_{nk}(na)$ are independent of the subscript $k$ and have the form

$$P[\xi_{nk}(na) = s] = (P(a) - N(a)) (G_n(s,a) + \alpha A_n(s-1,a) + (1-\alpha) A_n(s,a)) + N(a) (G_n(s+1,a) + \alpha A_n(s,a) + (1-\alpha) A_n(s+1,a)) + q(a) R_n(s,a),$$

where

$$G_n(s,a) = \int_0^\infty (1-F_{\theta_\alpha}(y)) e^{-a(y)} \frac{[\lambda(a)y]^y}{y!} dF_{\theta_\alpha}(y);$$

$$A_n(s,a) = \int_0^\infty (1-F_{\theta_\alpha}(y)) \int_0^\infty e^{-a(y)} \frac{[\lambda(a)x]^y}{y!} dF_{\theta_\alpha}(y);$$

$$R_n(s,a) = \int_0^\infty e^{-a(y)} \frac{[\lambda(a)x]^y}{y!} dF_{\theta_\alpha}(x).$$

Here $P(a) = (\lambda(a) + \alpha \gamma(a))(\lambda(a) + \alpha \gamma(a) + \mu(a))^{-1}; q(a) = 1 - P(a); m(a) = P(a)(m_Z + \beta m_U) + q(a)(m_V + \mu(a))^{-1}$.

**THEOREM 2.** If in addition to the conditions of Theorem 1 the functions $\lambda(\cdot), \nu(\cdot), \mu(\cdot)$ are continuously differentiable, $\theta_n^2, \nu_n^2, U_n^2, V_n^2$ are uniformly integrable, and for $n \to \infty$

$$\sqrt{n} (m_Z + m_U + m_V - m_Z - m_V - m_U) \to 0;$$

$$m_Z^2 \to m_Z^2, m_U^2 \to m_U^2, m_V^2 \to m_V^2; m^2 \to \frac{1}{\sqrt{n}} (Q(0) - \eta_0) = \xi_0,$$

then the sequence of measures generated by the processes $\zeta_n(t) = 1/\sqrt{n} (Q(n)t - \eta(t)), t \geq 0$, weakly converges in the space $D_T$ for every $T > 0$ to the measure generated by the process $\zeta(t)$ that satisfies the stochastic differential equation

$$d\zeta(t) = g(\zeta(t)) \zeta(t) dt + D(\zeta(t)) \eta(t) - \nu^2 dW(t),$$

where

$$g(a) = \frac{d}{da} [\lambda(a) - \nu(a) (1-\alpha) \mu(a)^{-1}];$$

$$D_t(n) = \lambda^2(n) [p(a) (m_Z^2 + \beta m_U^2 + 2 \beta m_V^2) + \nu(a) m_V^2] + \lambda(a) m(a) +$$

$$+ P(a) [2\lambda(a) (b+\beta m_U) + \beta] + N(a) [1 - \beta(1+\alpha + 2\lambda(a) m_Z) -$$

$$- (P(a) - N(a))(1+2\alpha \gamma(a)m_Z) - \lambda(a) m(a) -$$

$$- P(a) (1-\alpha \gamma)^2 (1-\sigma^2(a) m(a)^{-2})$$

and

$$\tau_{nk}(na) = \begin{cases} X_n + Z_n + U_n^2(\theta_n < x_n) & \text{with probability } P(a), \\ X_n + V_n & \text{with probability } q(a), \end{cases}$$

where $P(X_n > x) = \exp\{-\lambda(a) + \alpha \gamma(a) + \mu(a))x\}.$

Determining the numerical characteristics $\xi_{nk}(na)$ and $\tau_{nk}(na)$ and using Theorems 1 and 2 [4], we obtain the initial assertion. In particular, when $\lambda(a) = \lambda, \nu(a) = \nu, \mu(a) = \mu$, we easily obtain

$$\eta(t) = C + (\eta_0 - C) \exp\left[-\nu \frac{1 - \lambda(m_Z + \beta m_U) - \alpha \beta}{(1-\alpha \beta) (1+\mu m_U)} +$$

$$+ \frac{m_Z + \beta m_U}{(1-\alpha \beta) (1+\mu m_U)} \frac{1 - \lambda(m_Z + \beta m_U) - \alpha \beta}{(1-\alpha \beta) (1+\mu m_U)} (\eta_0 - \eta(t))\right],$$

where

$$C = \nu \frac{\lambda \lambda(m_Z + \beta m_U) + \alpha \beta + \mu m_U}{\nu [1 - \lambda(m_Z + \beta m_U) - \alpha \beta]}.$$

Note that when $\lambda(m_Z + \beta m_U) + \alpha \beta < 1$, a stationary operating mode of the system exists and for $t \to \infty$ and any $\eta_0$ we have $\eta(t) \to C.$