Chamfer Metrics in Mathematical Morphology*

P.F.M. NACKEN
TNO Human Factors Research Institute, Kampweg 5, 3769 DE Soesterberg, The Netherlands

Abstract. This paper presents an integration of chamfer metrics into mathematical morphology. Because chamfer metrics can approximate the Euclidean metric accurately, morphological operations based on chamfer metrics give a good approximation to morphological operations that use Euclidean discs as structuring elements. First, a formal definition of chamfer metrics is presented and some properties are discussed. Then, a number of morphological operations based on chamfer metrics are defined. These include the medial axis, the medial line, size and antisize distributions, and the opening transform. A theoretical analysis of some properties of these operators is provided. This analysis concentrates on the relation between distance transformations and reconstructions and the morphological operators just mentioned. This leads to a number of efficient algorithms for the computation of the morphological operators. All algorithms (except for the opening transform) require a fixed number of image scans and are based on local operations only. An algorithm for the opening transform that is 50–100 times as fast as the brute-force algorithm is presented.

Key words. Mathematical morphology, Chamfer metrics, algorithms

1 Introduction

This paper treats the relation between mathematical morphology [9], [16] and chamfer metrics [3]. It contains the analysis of properties of chamfer metrics and derives a description of morphological operators in terms of metrics and distance transforms. From this description a number of algorithms for morphological operators are derived. This paper is restricted to morphology for binary images.

The definition of morphological operators requires the choice of a structuring element, a small set that is used as a probe. In many cases mathematical morphology uses families of operators constructed by applying a simple operation, such as the opening, with structuring elements of increasing size. In the continuous case, for which images are subsets of the Euclidean plane, discs of increasing size are an appropriate choice for a family of structuring elements.

In practical situations image-processing systems perform their operations on images defined on a discrete square grid. A common choice for a family of structuring elements on the discrete grid is a family of the form $nB = B \oplus \cdots \oplus B$, where $B$ is a square or a diamond. The disadvantage of this choice is that such a family of structuring elements is quite dissimilar from the family of Euclidean discs. Because chamfer metrics are a good approximation of the Euclidean metric, spheres in the chamfer metric seem to be a suitable choice as a family of structuring elements. Such a choice, however, poses some other problems, for example, those caused by the fact that larger spheres are in general not invariant under an opening by smaller spheres.

The goal of this paper is the integration of chamfer metrics into mathematical morphology. Both the theoretical and practical aspects of such integration are discussed. The theoretical part consists of the construction of a number of morphological operators based on chamfer metrics and the analysis of such operators in terms of the metric. From this analysis a number of efficient algorithms are derived, providing the practical part of the integration.

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The paper starts with the description of some properties of general discrete metrics, a formal definition of the chamfer metric, and a discussion of some of its properties. Then, several morphological transformations, such as size distributions, antisize distributions, the medial axis, and the opening transform, are defined by using the chamfer metric and their properties are analyzed. This analysis leads to a number of efficient algorithms for performing these operators.

Efficient algorithms for morphological operations have been described by a number of authors. Groen and Foster [5] use lookup tables to speed up decisions based on inspection of the neighborhood of a pixel. Schmitt and Vincent [15], [20] use queues in which only pixels that must be processed are stored and pixels that do not need to be processed are ignored. Van den Boomgaard and van Balen [18] use a decomposition of the structuring element combined with a bit-mapped storage structure for the image to construct efficient algorithms. The algorithms described by these authors are not applicable to operations based on the chamfer metric.

The chamfer metric has also been used for the construction of skeletons of objects [4], [11], [13]. (Here, the phrase skeleton of an object refers to a thin set with the same homotopy as that of the object. In section 4 the different ways in which the terms skeleton and medial axis are used in literature will be discussed.) Authors using the chamfer metric for constructing skeletons usually compute skeletons by detecting ridges in the distance transform of an object or by a thinning algorithm in which pixels are scanned in order of increasing distance-transform value. Another approach of skeletonization is oriented more towards mathematical morphology and defines a skeleton as the locus of centers of maximal spheres in an object [16]. In this paper the detection of centers of maximal spheres and thinning are combined.

The organization of the rest of this paper is as follows. In section 2 discrete metrics are introduced and some other definitions are presented. Section 3 presents chamfer metrics and some of their properties. This section also provides algorithms for distance transforms and reconstructions for the chamfer metric, which are the building blocks for the algorithms to be presented in sections 4-6. Section 4 presents the medial axis and the construction of a homotopy-preserving medial line from the medial axis. Section 5 presents size distributions and antisize distributions. Section 6 presents the opening transform. Section 7 sums up the conclusions of this paper.

2 Discrete Metrics

In this section discrete metrics are defined and some of their properties are described.

A metric on a set $E$ is a function $d : E \times E \to [0, \infty]$ satisfying the following conditions:

1. $d(x, y) = 0 \iff x = y$ for all $x, y \in E$;
2. $d(x, y) = d(y, x)$ for all $x, y \in E$;
3. $d(x, y) + d(y, z) \leq d(x, z)$ for all $x, y, z \in E$.

Note that in this paper metrics are allowed to assume the value $\infty$. This is necessary because later on metrics will be constructed for which the distance between two points is defined as the length of a shortest path between them. If there is no path between two points, the distance between these points is $\infty$. For chamfer metrics, which are a special case of shortest-path metrics, all distances are finite.

Let $d$ be a metric on a set $E$. Let $D$ denote the set $\{d(x, y) \mid x, y \in E\}$; this set is called the range of the metric. For each $d \in D$ open and closed spheres can be defined.

**Definition 2.1.** Let $d$ be a metric on a set $E$, and let $D$ be its range. Let $r \in D$ and $x \in E$. The closed sphere with radius $r$ and center $x$ is the set

$$B(x, r) = \{y \in E \mid d(x, y) \leq r\}.$$

**Definition 2.2.** Let $d$ be a metric on a set $E$, and let $D$ be its range. Let $r \in D$ and $x \in E$. The open sphere with radius $r$ and center $x$ is the set

$$B(x, r) = \{y \in E \mid d(x, y) < r\}.$$