A HOMOCLINIC VERSION OF THE CENTRAL LIMIT
THEOREM

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The definitions of homoclinic partitions and transformations are given in situations that are standard for
topological dynamics and ergodic theory. A variant of the central limit theorem is proved, the formulation of
which makes use of homoclinic transformations.

INTRODUCTION

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, let \( T \) be an automorphism of it (i.e., a measurable invertible transformation
with invariant measure \( \mathbb{P} \)), and let \( f : \Omega \to \mathbb{R}^1 \) be a measurable function. In this paper we are interested in the conditions under
which the central limit theorem (CLT) can be applied to the sums \( \sum_{k \geq 0} f \circ T^k \). There arises at once the problem of finding
the terms in which these conditions must be expressed. The most attractive would be a "coordinate-free" variant of such condi-
tions, i.e., a variant when all objects, occurring in formulation of CLT, are constructed in terms of \( f \) and \( T \) in a canonical
manner. Presently, the author is not aware of such formulations (if we do not consider versions clause to tautologies, which,
for example, can be obtained by imposing appropriate conditions on the moments of the sequence \( \{ f \circ T^k, k \in \mathbb{Z} \} \). It is
unlikely that the problem would be essentially easier if we would restrict ourselves only to automorphisms \( T \) that are
isomorphic, let us say, to the Bernoulli shifts, without assuming that an isomorphism of this kind is fixed.

By virtue of this, presently, for the creation of circumstances in which one could formulate some variant of a CLT it
is necessary to fix beforehand some variant of a complementary structure. The most prevalent form of such a structure is some
distinguished family of \( \sigma \)-algebras, contained in \( \mathcal{F} \). Exactly in terms of such families one defines properties as the Markov
property, various intermixing variants, the martingale property. Moreover, as a rule, limit theorems for individual classes of
smooth dynamical systems are proved by the construction of special (Markov) partitions, possessing definite intermixing
properties, with the subsequent application of already purely probabilistic results [1]. The desire to shorten this sufficiently
tedious path and to avoid the use of such noncanonical objects as Markov partitions, has been the motivation for attempting
the search of an alternate approach, reflected in this paper.

It should be remarked that, although the \( \sigma \)-algebras are the most commonly used complementary structures applied in
connection with the CLT, one should not omit to mention the situation when \( \Omega \) has the structure of a group or of a
homogeneous space, which gives the possibility to use harmonic analysis. For the case when \( \Omega \) is a compact Abelian group,
while \( T \) is its ergodic automorphism (or even a surjective endomorphism), very general results have been obtained in its time
in [2] exactly along this path.

In this paper we attempt to apply to the topic of limit theorems complementary structures of another type, the so-called
homoclinic structures. If \( (X, d) \) is a metric space, \( \rho : X \to X \) a continuous mapping, \( x \) a fixed point of \( \rho \) (\( \rho(x) = x \)), then
\( y \in X \) is called a homoclinic point for \( x \) if \( \rho^n(y) \to x \) for \( |n| \to \infty \) (a more traditional definition, going back to Poincaré,
refers to a smooth situation and requires the hyperbolicity of the fixed point \( x \); in this case by the homoclinic points for \( x \) one
means the points of intersection, other than \( x \), of the stable and unstable layers, passing through \( x \)). If on \( X \) there is given also
a probability measure, invariant relative to \( \rho \), then in many cases the fixed and homoclinic points are concentrated on a set of
measure zero and do not present interest from the point of view of ergodic or probability theory. Much more appropriate for
investigation by the methods of these disciplines is the following object, to be called, naturally, homoclinic partition.

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By definition, the points $x_1, x_2 \in X$ belong to the same element of the homoclinic partition of the space $X$ (denoted $x_1 \sim x_2$) if $d(\rho^n(x_1), \rho^n(x_2)) \to 0$ for $|n| \to \infty$. The homoclinic partition is an object associated in a canonical manner with the triple $(X, d, \rho)$. If $X$ is compact, then it does not depend on the selection of the metric. The fact that this partition need not be a trivial one can be seen on the example of the situation where $X$ is a closed smooth manifold, while $\rho$ is an Anosov diffeomorphism (regarding information on Anosov diffeomorphisms, see [3]). In this case $x_1 \sim x_2$ if and only if $x_1$ and $x_2$ lie in the same fiber of the unstable fibration as well as in the same fiber of the stable fibration. In many examples each of the elements of the partition is dense in $X$, while the measurable hull of the partition is trivial (i.e., just as for the trajectory partition of an ergodic automorphism, there are no nonconstant measurable functions that are constant on the elements of the partition). It is very probable that this is a general property of the Anosov diffeomorphisms. Another source of examples is given by topological Markov chains. In this case $X$ is the space of two-sided sequences and two sequences belong to the same element of the homoclinic partition if and only if their terms differ only for a finite number of indices.

The homoclinic partition is, from the point of view of a topological or smooth dynamics, an object that is canonically connected with the initial dynamical system and can be an important means for the investigation of such a system, in particular, for the construction of its invariants. If such a dynamical system possesses an invariant probability measure, then the automorphism of the probability space, arising in the described situation, turns out to be endowed with a complementary structure of an invariant partition. This partition belongs to a class of partitions, called in [4] measurable equivalence relations, while in V. A. Rokhlin's seminar have been called at that time nonmeasurable. The first term is justified by the measurability of the graph of this partition, while the second one by the absence, in the general case, of a reasonable measurable structure on the quotient space with respect to this partition.

If a probability space with automorphism is devoid of any supplementary structure, then it is unlikely that in such a situation one can define by a canonical construction a partition playing the role of a homoclinic partition. Moreover, it is not entirely clear how the definition of such a partition must look under such circumstances. In addition, even in those cases when the homoclinic partition is inherited through the automorphism of the probability space from some richer structure, so far one has not succeeded to prove the CLT by making use only of the homoclinic partition. All this constitutes the basis for introducing the following definition.

**Definition.** A measurable, invertible, nonsingular transformation $S$ of a space $(\Omega, \mathcal{F}, P)$ is said to be a homoclinic transformation for an automorphism $T$ if $T^nST^{-n} \to I$ for $|n| \to \infty$ (here $I$ is the identity mapping, while the convergence of transformations is understood as the weak convergence of the operators induced by them in $L^2$).

**Remark 1.** The transformation $T$ acts by the conjugation $S \to TST^{-1}$ on the group of invertible nonsingular transformations, while $I$ is a fixed point of this action. The points that are homoclinic for $I$ in this infinite-dimensional dynamical system coincide with the transformations that are homoclinic for $T$ in the sense of the above formulated definition.

**Remark 2.** In the case when $S$ preserves the measure $P$, an equivalent definition is obtained if we require that the operators induced in $L^2$ by the transformations $T^nST^{-n}$ should converge weakly (or strongly).

**Remark 3.** Similar definitions can be given in the metric and the differentiable cases.

**Remark 4.** If $X$ is a compact metric space with homeomorphism $\rho: X \to X$, then the transformations that are homoclinic for $\rho$ form a group, each orbit of which is included in one of the elements of the homoclinic partition. It is not known whether this inclusion can be strict. The situation is analogous with the closed differentiable manifolds and their diffeomorphisms. However, in the case to which the above given definition refers, it is not even clear whether the product of two homoclinic transformations is homoclinic.

**Remark 5.** Similar definitions can be given also when instead of the action of the group $\mathbb{Z}$, defined by the automorphism $T$, one considers the action of an arbitrary locally compact group. This enables us to apply homoclinic methods to the investigation of random fields.

The usefulness of homoclinic transformations (at least of those that preserve $P$) for the proof of limit theorems can be elucidated by the fact that the sequences $\{ f \circ T^k, k \in \mathbb{Z} \}$ and $\{ f \circ (T^kS), k \in \mathbb{Z} \}$ come together asymptotically for $|k| \to \infty$ (in the sense of the $L^2$-metric, if $S$ preserves the measure $P$). This means that $S$ enables us to "deform" the sequence $\{ f \circ T^k, k \in \mathbb{Z} \}$, where the effect of the deformation is damped as we move away from some finite subset of $\mathbb{Z}$. The possibility of such a deformation can be considered as the manifestation of the fact that the peripheral values of the sequence depend weakly on its values on some bounded set. For random processes, possessing some property of weak dependence in the traditional sense (for example, some intermixing property), one can consider $\sigma$-algebras $\mathcal{F}_A$, $A \in \mathbb{Z}$, generated by the values of the process, related to moments and belonging to the complement to some finite set $A$. It is natural to call such $\sigma$-algebras peripheral. Their intersection over all $A$... is usually trivial, but, as a rule, the partition that is the limit