Weighted Majority Games Have Many $\mu$-values$^1$

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1 Introduction

Measure-based values been introduced by Aumann and Kurz ([1], 1977) and have been discussed by Hart ([3], 1980), Monderer ([4], 1986), and Monderer and Neyman ([5], 1988).

Let $\mu$ a non-atomic probability measure. It was proved in [4] that the $\mu$-symmetry axiom is sufficient (together with the linearity and the efficiency axioms) to determine the Aumann and Shapley value on the space $pN A(\mu)$ of smooth non-atomic games which are absolutely continuous with respect to $\mu$. A shorter proof of this fact was given in [5].

In this paper we show that this is not the case when we move away from smooth games. It is proved that there are many $\mu$-values on the space $bv'N A(\mu)$, which is the closed linear space generated by the smooth games, and by all weighted majority games that are continuous at $\emptyset$ and at $I$ (the set of players) with weights which are absolutely continuous with respect to $\mu$.

Since every $\mu$-value on $bv'N A(\mu)$ is continuous with norm 1 and satisfies the projection axiom, and because by [6] $bv'N A(\mu)$ is a subspace of $ASY MP$, we provide an example of a $\mu$-value of norm 1 which is not a partition $\mu$-value, where the notion of a partition $\mu$-value is defined in analogy to the notion of a partition value defined in [7]. It is not known whether there is a value with norm 1 which is not a partition value.

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2 \( \mu \)-Values on \( bv'NA(\mu) \)

We shall use the same notations as in [2] and [4] and also the following:

\( bv'NA(\mu) \) will denote the closed linear subspace of \( BV \) generated by all games of the form \( v = f \circ \lambda \), where \( \lambda \in NA^1(\mu) \) (i.e., \( \lambda \) is absolutely continuous with respect to \( \mu \)) and \( f \in bv' \) (i.e., \( f \) is a function of bounded variation on the interval \( [0, 1] \), which is continuous at 0 and 1 and satisfies \( f(0) = 0 \)). \( s'NA(\mu) \) will denote the closed linear subspace of \( bv'NA(\mu) \) generated by all games of the form \( v = f \circ \lambda \), where \( \lambda \in NA^1(\mu) \) and \( f \in s' \) (i.e., \( f \) is a singular function in \( bv' \), that is \( f'(x) = 0 \) a.e. w.r.t. the Lebesgue measure). Finally, let \( Q \) be the dense subspace of \( s'NA(\mu) \) consisting of all games of the form \( v = \sum_{i=1}^{n} f_i \circ \lambda_i \), where \( f_i \in s' \) and \( \lambda_i \in NA^1(\mu) \).

**Theorem A:** There exist continuous \( \mu \)-values with norm 1 on \( bv'NA(\mu) \) which satisfy the projection axiom and are different from the restriction to \( bv'NA \) of the Aumann-Shapley value on \( bv'NA \).

Before proving Theorem A we state one of its corollaries whose proof is given in the introduction.

**Corollary 1:** There exists a continuous \( \mu \)-value with norm 1 which satisfies the projection axiom and which is not a partition value.

The proof of Theorem A will be given through Lemmas 2–4.

**Lemma 2:** There exists a function \( \tau : NA^1(\mu) \rightarrow NA^1(\mu) \), different from the identity map, such that:

1. \( \tau \) is \( \mu \)-symmetric. i.e.,
\[
\tau(\theta_\ast \circ \lambda) = \theta_\ast \circ \tau(\lambda)
\]
for every \( \lambda \in NA^1(\mu) \) and every \( \theta \in G(\mu) \),

where \( G(\mu) \) is the group of all automorphisms of the players set \( (I, \mathcal{C}) \) which preserve \( \mu \).

2. \( \tau \) satisfies the dummy property. i.e.,
\[
\tau(\lambda) \text{ is absolutely continuous w.r.t. } \lambda
\]
for every \( \lambda \in NA^1(\mu) \).

**Proof:** Let \( T \) be any measurable subset of \( I \) for which \( 0 < \mu(T) < 1 \). Define measures \( \mu_1, \mu_2 \) and \( \mu_0 \) as follows:
\[
\mu_1(S) = \frac{\mu(S \cap T)}{\mu(T)} ; \quad \mu_2(S) = \frac{\mu(S \cap T^c)}{\mu(T^c)} ;
\]