A FACTORIZATION OF REGULAR GENERALIZED NEVANLINNA FUNCTIONS

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Let $Q$ be a regular operator valued generalized Nevanlinna function with negative index $\kappa$, i.e. $Q \in \mathcal{N}_\kappa(\mathcal{H})$. It is shown that then there exists a rational function $B(z)$, which collects the generalized poles and zeros of $Q$ that are not of positive type such that the function

$$B(\bar{z})^*Q(z)B(z)$$

belongs to the Nevanlinna class $\mathcal{N}_0(\mathcal{H})$.

1 Introduction and preliminaries

Let $(\mathcal{H}, (\cdot, \cdot))$ be a Hilbert space. Recall that an operator function $Q$ with values in $\mathcal{L}(\mathcal{H})$ belongs by definition to the generalized Nevanlinna class $\mathcal{N}_\kappa(\mathcal{H})$, if it is meromorphic in $\mathbb{C} \setminus \mathbb{R}$, symmetric with respect to the real axis (that is $Q(\bar{z})^* = Q(z)$ for $z \in \mathcal{D}$, the domain of holomorphy of $Q$), and if the so-called Nevanlinna kernel

$$N_Q(z, \zeta) = \frac{Q(z) - Q(\zeta)^*}{z - \overline{\zeta}}, \quad z, \zeta \in \mathcal{D} \cap \mathbb{C}^+$$

has $\kappa$ negative squares. This means that for arbitrary $N \in \mathbb{N}$, $z_1, \ldots, z_N \in \mathcal{D} \cap \mathbb{C}^+$ and $\tilde{\xi}_1, \ldots, \tilde{\xi}_N \in \mathcal{H}$ the Hermitian matrix

$$\left( N_Q(z_i, z_j) \tilde{\xi}_i, \tilde{\xi}_j \right)_{i,j=1}^N$$

has at most $\kappa$ negative eigenvalues, and $\kappa$ is minimal with this property. It is well known (see e.g. [KL 77], [HSW]) that the class $\mathcal{N}_\kappa(\mathcal{H})$ contains exactly those operator functions which admit a minimal representation of the form

$$Q(z) = Q_0 + (z - z_0) \Gamma^+ \left( I + (z - z_0)(A - z)^{-1} \right) \Gamma$$

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with some Pontryagin space \((\mathcal{K}, [\cdot, \cdot])\) with negative index \(\kappa\). Here \(\Gamma: \mathcal{H} \rightarrow \mathcal{K}\) is a bounded linear operator. By \(\Gamma^+\) we denote the adjoint of \(\Gamma\), defined by \([\Gamma \varphi, x]_\mathcal{K} = (\varphi, \Gamma^+ x)_\mathcal{H}\) for all \(x \in \mathcal{K}\), \(\varphi \in \mathcal{H}\). The linear relation \(A\) in \(\mathcal{K}\) is self-adjoint with non-empty resolvent set. The point \(z_0 \in \varrho(A) \cap \mathbb{C}^+\) is the fixed point of reference. For the operator \(Q_0 \in \mathcal{L}(\mathcal{H})\) and its adjoint \(Q_0^*\) it holds

\[
Q_0 - Q_0^* = (z_0 - \overline{z_0})\Gamma^+ \Gamma. \tag{1.2}
\]

The representation (1.1) is called \textit{minimal} if

\[
\mathcal{K} = \text{c.l.s.} \left\{ (I + (z - z_0)(A - z)^{-1})\Gamma \varphi \big| z \in \varrho(A), \varphi \in \mathcal{H} \right\},
\]

where c.l.s. stands for closed linear span. In fact, because of the holomorphy of the resolvent operator \((A - z)^{-1}\), it is sufficient to consider \(z\) in some subset \(\Phi \subseteq \varrho(A)\), which contains in each component of \(\varrho(A)\) an open non-empty subset.

We call the function \(Q \in \mathcal{N}_\kappa(\mathcal{H})\) \textit{regular} if its domain of holomorphy contains at least one point \(w_0\), such that the operator \(Q(w_0)^{-1}\) is boundedly invertible. That is, \(Q(w_0)^{-1}\) exists as an operator and is defined on the whole space \(\mathcal{H}\) (and is hence bounded). In the case that the function \(Q\) is regular, without loss of generality, we may assume that \(Q_0 = Q(z_0)\) is boundedly invertible. With a regular function \(Q \in \mathcal{N}_\kappa(\mathcal{H})\) also the "inverse" function

\[
\hat{Q}(z) := -Q(z)^{-1}
\]

belongs to the generalized Nevanlinna class \(\mathcal{N}_\kappa(\mathcal{H})\). In Section 2 we give a representation for the function \(\hat{Q}\) by means of the representation of \(Q\).

The point \(\alpha \in \mathbb{C} \cup \{\infty\}\) is called a \textit{generalized pole} of \(Q \in \mathcal{N}_\kappa(\mathcal{H})\) if \(\alpha\) is an eigenvalue of the relation \(A\) in the representation (1.1). If \(x_\alpha\) is an eigenvector of \(A\) at \(\alpha \in \mathbb{C}\) (at \(\alpha = \infty\)), then the vector

\[
\tilde{n}_\alpha := (\alpha - \overline{z_0})\Gamma^+ x_\alpha, \quad (\tilde{n}_\alpha := \Gamma^+ x_{\infty}, \text{respectively}) \tag{1.3}
\]

is called a \textit{pole vector} of \(Q\) at \(\alpha\). The pole vector \(\tilde{n}_\alpha\) is called \textit{positive} (\textit{negative}, \textit{neutral}), if the eigenvector \(x_\alpha\) in (1.3) is positive (negative or neutral). A generalized pole \(\alpha\) is called of \textit{positive type} if the eigenspace of \(A\) at \(\alpha\) (and hence also every pole vector at \(\alpha\)) is positive. Note that every non-real pole of \(Q\) is an eigenvalue of \(A\) with a corresponding neutral rootspace, whereas a pole vector at a real (generalized) pole can be of any type. The set of generalized poles, furthermore, can also contain points that are not isolated singularities of the function \(Q\). Since the representation (1.1) of \(Q\) is assumed to be minimal, \(\tilde{n}_\alpha \neq 0\) and, furthermore, the type of a pole vector is well defined.

\textbf{Remark.} For \(\dim \mathcal{H} < \infty\) these points have been characterized also analytically by the existence of a so-called pole function \(\tilde{n}(z)\) (see [BL] and [DLS]). It holds

\[
\lim_{z \to \infty} \tilde{n}(z) = \tilde{n}_0,
\]

where \(\tilde{n}_0\) is a pole vector of \(Q\) at \(\alpha\).