For designing an isotropic inhomogeneous body having a variable elastic modulus and a prescribed shape under the influence of an external force load we give a formulation and solution of the problem of determining the design that is optimal with respect to stress. The problem of optimal design reduces to a certain problem in the theory of elasticity for a nonlinear-elastic material. As an example we consider the problem of optimal design of an inhomogeneous cylinder. Four figures. Bibliography: 9 titles.

Many modern methods of obtaining and processing materials make it possible to alter their elastic parameters within rather wide limits. In this connection it is important to study the problems of determining the elastic inhomogeneity that optimizes the stress-deformed state. The problem of the distribution of elastic parameters in an isotropic body from the condition that the work of external forces is minimized has been studied by Lur'e [6]; the problem of the distribution of the shear modulus that optimizes the rigidity of a rod under torsion has been studied by Lavrov, Lur'e, and Cherkaev [5], and the solution of the problem of the distribution of elastic parameters in a sphere for a minimizing displacement of the outer surface is given in [9].

In the present paper we obtain the solution of the problem of finding an elastic inhomogeneity in an isotropic body from the condition that the stress level be minimized [3].

Consider a linear elastic inhomogeneous isotropic body occupying a finite region $V$ with a piecewise-smooth boundary, part of which is held fast. The body is in equilibrium under the action of the surface forces given on the rest of the surface and the three-dimensional forces. Many structural metals [2] have significantly different Young's moduli $E$, yet nearly identical Poisson coefficients $\nu$. In this connection we consider the problem of choosing a law of variation of the Young's modulus on $V$ with a fixed Poisson coefficient in such a way as to minimize the stress level, in the form

$$\|\sigma\| = \max_V \rho g^{\frac{1}{2}}(\sigma) \to \min_E, \quad g(\sigma) = \frac{1}{3}(1 - 2\nu)\sigma_{kk}\sigma_{nn} + (1 + \nu)s_{ij}s_{ij}. \quad (1)$$

Here $\rho$ is a given strictly positive weight function and $s$ is the deviator of the stress tensor $\sigma$.

We use the well-known method [1] of replacing the local quality criterion by an integral criterion and consider a stress level of the form

$$\|\sigma\|_p = f_p(g), \quad f_p(g) = \left[ V^{-1} \int_V (\rho^2 g)^p dV \right]^{\frac{1}{p}}, \quad (2)$$

which tends to the value (1) of the local stress level as the exponent $p$ increases.

Consider the problem of choosing a piecewise-continuous distribution of the Young's modulus on $V$ that minimizes the stress level in the given body:

$$\|\sigma_p\| \to \min_E \quad (3)$$

with a given shape of the body and given external loads and given fastening conditions. We shall assume in what follows that there exists an optimal stressed state and a design $E_*$ for it.

We shall show that the optimal distribution of the Young's modulus is connected with the stress field by the following optimality condition:

$$E_*^{-1} = \text{const}^2 \cdot \rho^{2p} g^{p-1}(\sigma^*). \quad (4)$$

We shall also show that \( \sigma^* \) is the solution of the following problem of minimizing the level of stress among the statically possible stress fields
\[
\|\sigma\|_p \rightarrow \min, \quad \sigma \in \sigma^0 + \Psi.
\] (5)
Here \( \sigma^0 \) is the particular solution of the inhomogeneous equilibrium equations in the stresses that satisfies the inhomogeneous static boundary conditions; \( \Psi \) is the space of stress fields that satisfy the homogeneous equilibrium equations with homogeneous force boundary conditions.

We shall show that condition (4) is a sufficient condition for a global optimum in the problem (3). Suppose the design \( E_* \) realizes the stress field \( \sigma^* \) in the given elastic body; then by Castigliano's principle [7]
\[
K(\sigma^*) = \min_{\sigma \in \sigma^0 + \Psi} K(\sigma), \quad K(\sigma) = \int_V E_*^{-1} g(\sigma) \, dV,
\] (6)
\[
\int_V E_*^{-1} \left[ \frac{1}{3} (1 - 2\nu) \sigma_{kk} \hat{s}_{nn} + (1 + \nu) s_{ij}^* \delta_{ij} \right] dV = 0, \quad \forall \hat{\sigma} \in \Psi.
\]
Suppose that condition (4) holds. Then, substituting it into (6), we obtain a relation for the stress field \( \sigma^* \) that coincides with the necessary condition for optimality in the problem (5). Computing the second variation \( \|\sigma\|_p \), it is not difficult to verify that it is strictly positive definite. Consequently the problem (5) reduces to minimizing a strictly convex function \( \|\sigma\|_p \) on the convex (here affine) set \( \sigma^0 + \Psi \); therefore the solution \( \sigma^* \) is unique by a well-known property [8] of convex programming. Thus if \( E_* \) and \( \sigma^* \) satisfy the optimality condition (4), then \( \sigma^* \) is a solution of problem (5), i.e., it has a minimal stress level among the statically possible stress fields. Since all stress fields that are achievable in the given body by choosing a design are statically admissible, the solution \( E_*, \sigma^* \) just found is globally optimal in the original optimal design problem (3). The sufficiency of the optimality conditions (4) has already been shown. The necessity of these conditions can be proved by the method described in [3].

Thus the solution of the problem (3) of optimizing the stressed state in an inhomogeneous isotropic elastic body has been reduced to solving a physically nonlinear problem of the theory of elasticity in which the Young's modulus is determined from the stresses via the law (4). The solution of this problem can be obtained as follows: first the stresses \( \sigma^* \) are determined from problem (5); then the unknown distribution \( E_* \) is found from (4) up to an arbitrary positive constant factor. This arbitrariness reflects the general property of invariance of stresses under a proportional change in the distribution of the elastic parameters of the body. The problem of the properties of different designs that realize the same stress field is studied in more detail in [4].

As an example we consider the problem of determining the Young's modulus \( E(r) \) along the radius of an inhomogeneous circular cylinder in conditions of axisymmetric strain under the influence of a constant internal pressure \( p_1 \) and a constant external pressure \( p_2 \). It is required to minimize the stress level (2):
\[
\|\sigma\|_p = \left[ \frac{1}{2} (R_2^2 - R_1^2)^{-1} \int_{R_1}^{R_2} g^p r \, dr \right]^{\frac{1}{2}}, \quad g = (1 + \nu)[(\sigma_r - \sigma_\theta)^2 + (1 - 2\nu)(\sigma_r + \sigma_\theta)^2]/2.
\] (7)
Here \( \sigma_r, \sigma_\theta, R_1, \) and \( R_2 \) are respectively the radial and circumferential stresses and the inner and outer radii of the cylinder; the weight function \( \rho \) is assumed equal to 1.

The relations of the linear theory of elasticity, supplemented by the optimality condition (4), comprise a closed system of equations:
\[
\frac{du}{dr} + \frac{u}{r} = \frac{1}{2} \delta(3 - \delta)(\sigma_r + \sigma_\theta)B, \quad \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad \frac{du}{dr} - \frac{u}{r} = \frac{1}{2} (3 - \delta)(\sigma_r - \sigma_\theta)B, \quad B = k^2 g^{p-1},
\] (8)
\[
g = \frac{1}{4} (3 - \delta)[(\sigma_r - \sigma_\theta)^2 + \delta(\sigma_r + \sigma_\theta)^2],
\]
with the boundary conditions
\[
\sigma_r(R_1) = -p_1, \quad \sigma_r(R_2) = -p_2,
\] (9)