On Existence and Behaviour of Asymptotically Flat Solutions to the Stationary Einstein Equations

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Abstract. We show existence and uniqueness of asymptotically flat solutions to the stationary Einstein equations in $S = \mathbb{R}^3 - B_r$, where $B_r$ is a ball of radius $r > 0$, when a small enough continuous complex function $\tilde{u}$ on $\partial S$ is given. Regularity and decay estimates imply that these solutions are analytic in the interior of $S$ and also at infinity, when suitably conformally rescaled.

Introduction

After a considerable effort a rather clear picture describing the set of stationary, and asymptotically flat vacuum solutions to the Einstein equations is now available. Solutions are represented by a complex scalar field $\tilde{u}$, and a positive definite metric $g_{ab}$ on a three dimensional manifold $S$ [1–3]. From the four dimensional point of view this manifold is the quotient of space-time with the set of orbits of the Killing vector field defining stationarity, the metric is conformally related to the one induced by the space-time metric on $S$, and $\tilde{u}$ is a given functional of the norm, and the twist of the Killing vector field. The equations they satisfy are,

\begin{align}
(A_g - 2R)\tilde{u} &= 0, \\
G_{ab} - 2(V_a \tilde{u} V_b \tilde{u}^* - (1 + 4|\tilde{u}|^2)^{-1/2} V_a |\tilde{u}|^2 V_b |\tilde{u}|^2) \\
- g_{ab}(P_c \tilde{u} P_c \tilde{u}^* - (1 + 4|\tilde{u}|^2)^{-1/2} P_c |\tilde{u}|^2 P_c |\tilde{u}|^2) &= 0,
\end{align}

where $G_{ab}$ is the three dimensional Einstein tensor corresponding to $g_{ab}$. We consider the following asymptotic boundary conditions $\tilde{u} \to 0$, $g_{ab} - \epsilon_{ab} \to 0$, as $r \to 0$, where $\epsilon_{ab}$ is any flat metric on $S$, and $r$ is the distance function with respect to it.

Given a solution $(\tilde{u}, g_{ab})$ of the above equations it is possible to reconstruct a unique, stationary, asymptotically flat, maximally extended vacuum space-time.

From local elliptic theory [4, 5] we know that sufficiently smooth solutions (if they exist) are in fact analytic. Furthermore assuming a stronger asymptotic decay than the one above, one can show there exists a conformal factor such that the conformally rescaled fields are also sufficiently smooth and satisfy regular elliptic equations; thus they are also analytic, even at the point representing infinity.
It makes sense then to characterize these solutions by their Taylor expansion at infinity, that is by a set of multipole moments \([1, 2, 8, 9]\). It turns out that to characterize these solutions it is enough to define multipole moments only for the conformally rescaled field corresponding to \(u\), that is there are no degrees of freedom associated to the three-metric \(g_{ab}\) \([10,11]\).

Unfortunately only a small set of solutions to the above equations, all of them possessing extra symmetries, are explicitly known and so until now we did not know if the asymptotic conditions assumed to obtain the above picture were generic enough so as to allow for the existence of a sufficiently large class of solutions. In other words we did not know how generic was this picture. In order to add weight to the above picture we shall show here the existence of a large class of solutions to the stationary Einstein equations when \(S = \{R^3 - \text{a smooth ball}\}\). These solutions, which one would like to interpret as the exterior field of stationary compact bodies, are analytic in the interior of \(S\), (even at infinity when suitably conformally transformed) and are uniquely characterized by a continuous complex function \(\hat{u}\) on \(\partial S\), the value of \(u\) at \(\partial S\), and thus by a harmonic expansion of \(u\) at \(\partial S\). To each member of this harmonic expansion there corresponds a unique linearized multipole moment at infinity.

**Main Theorem**

**Existence Theorem.** Fix on \(\partial S\) a positive definite metric, \(h_{ab}\), of constant scalar curvature. Then there exists a neighborhood of zero, \(V\), in the Sobolev space\(^2\) \(H^2(\partial S)\), such that for each \(\hat{u} \in V\) there exists a unique, analytic on \(\text{int} S\), and asymptotically flat solution to the stationary vacuum Einstein equations, \((u, g_{ab})\), with \(u|_{\partial S} = \hat{u}\). The solutions, when suitably conformally transformed are analytic even at infinity, and so each of them determines a unique multipole expansion at the point at infinity.

We partition the proof in a series of lemmata. In the first we show that the stationary equations are equivalent to a reduced elliptic system. This procedure is similar to the one used to reduce the full Einstein equations to a hyperbolic one \([12-14]\).

**Reduction Lemma.** Fix a flat metric, \(e_{ab}\), in \(S\) in such a way that it induces \(h_{ab}\) on \(\partial S\). Then the stationary vacuum Einstein equations, Eqs. \((1,2)\), with the boundary condition, \(u|_{\partial S} = \hat{u}\), for \((u, g_{ab} - e_{ab}) \in H^{3/2}_{S - 3/4}(S)^3\), \(\hat{u} \in H^2(\partial S)\), small enough, are equivalent to the following reduced system for \((u, \phi^{ab})\):

\[
E(u, \phi^{ab}) := \Delta u + (\text{terms in } \nabla \phi^{de}, \phi^{de}, \nabla u, u) = 0, \\
E^{ab}(u, \phi^{ab}) := g^{cd} \nabla_c \nabla_d \phi^{ab} + (\text{terms in } \nabla \phi^{de}, \phi^{de}, \nabla u, u) = 0, \\
\frac{C\psi}{(s - \sigma)} = 0, \quad \phi|_{\partial S} = 0, \quad \sigma|_{\partial S} = 0, \\
\psi|_{\partial S} = 0, \quad \psi^{\partial S} = 0,
\]

\(^1\) For a sufficiently large class of solutions we mean one containing all solutions corresponding to physical bodies in equilibrium occupying a compact region of space.

\(^2\) Proper definitions of all the spaces we are using, as well as a list of their properties can be found in \([15]\).

\(^3\) The differentiability and decay indices used for the weighted Sobolev spaces are not necessarily the sharpest ones, but rather are taken for definiteness and convenience.