ON THE STRUCTURE OF FINITE ELATION LAGUERRE PLANES

Dedicated to Professor W. Benz on the occasion of his 60th birthday

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The miquelian Laguerre plane of order \( q \) (\( q \) being a prime power) is obtained as the geometry of non-trivial plane sections of a quadratic cone in the 3-dimensional projective space over \( \text{GF}(q) \). Similarly, an ovoidal Laguerre plane of order \( q \) is obtained as the geometry of non-trivial plane sections of a cone over an oval (not necessarily a conic) in the 3-dimensional projective space over \( \text{GF}(q) \).

In general, a Laguerre plane \( \mathcal{L}=(\mathcal{P},\mathcal{X},\parallel) \) consists of a set of points \( \mathcal{P} \), a set of circles \( \mathcal{X} \) (considered as subsets of \( \mathcal{P} \)) and an equivalence relation \( \parallel \) on \( \mathcal{P} \) (parallelism) such that the following axioms hold (two points \( p,q \in \mathcal{P} \) are called parallel if and only if they are in relation \( p\parallel q \); otherwise they are called non-parallel):

(L1) Any three pairwise non-parallel points can be joined uniquely by a circle passing through these points.

(L2) To every circle \( K \) and any two non-parallel points \( p,q \) where \( p \in K \) and \( q \notin K \) there is precisely one circle \( L \) which is tangential to \( K \) at \( p \) (i.e. \( K \cap L=\{p\} \)) and passes through \( q \).

(L3) Every parallel class intersects each circle in a unique point.

(L4) There are at least two circles and each circle contains at least three points.

If \( \mathcal{P} \) is finite, any two circles have the same number \( n+1 \) of points, and \( n \) is called the order of \( \mathcal{L} \). There are \( n^2+n \) points, \( n^3 \) circles, and \( n+1 \) parallel classes in a

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Laguerre plane of order n, and every parallel class contains n points.

For every point $p \in P$ there is an internal incidence structure, whose point set consists of all points of $P$ not parallel to $p$ and whose set of lines consists of all circles containing $p$ (without the point $p$), and all parallel classes not passing through $p$; this is an affine plane, the derived affine plane $\mathcal{A}_p$ at $p$. We call the projective closure of $\mathcal{A}_p$ the derived projective plane $\mathcal{P}_p$ at $p$. If $\mathcal{L}$ has order $n$ then the derived affine plane and derived projective plane also has order $n$. A circle $K$ not passing through $p$ induces an oval in $\mathcal{P}_p$ by $(K \setminus (p^*)) \cup \omega$, where $p^*$ and $\omega$ denote the unique point on $K$ parallel to $p$ (axiom (L3)) and the infinite point of lines that come from parallel classes respectively; in particular, the infinite line of $\mathcal{P}_p$ (with respect to $\mathcal{A}_p$) is a tangent to this oval at $\omega$.

According to the celebrated theorem of Segre [19] an oval in a finite desarguesian projective plane of odd order is a conic. Chen and Kaerlein proved in [6] by simply counting the conics having a given tangent at a given point that a finite Laguerre plane of odd order having at least one desarguesian derived projective plane is miquelian. In particular, a Laguerre plane of (odd) order $< 7$ is miquelian. There are precisely two non-isomorphic Laguerre planes of order 8, the miquelian Laguerre plane and the ovoidal plane over a translation oval (not a conic). In [21] it was shown that also a Laguerre plane of order 9 is miquelian. At present there are no non-miquelian Laguerre planes of odd order known, and all known Laguerre planes of even order are ovoidal.

An automorphism of a Laguerre plane $\mathcal{L}$ is a bijection of the point set that maps circles onto circles. All automorphisms of $\mathcal{L}$ form a group $\Gamma$ with respect to composition, the automorphism group of $\mathcal{L}$. As every automorphism maps parallel points onto parallel points, the set of all automorphisms that map each point onto a parallel one forms a normal subgroup $T$ of $\Gamma$. Since $T$ is precisely the kernel of the action of $\Gamma$ on the set of parallel classes, we call $T$ the kernel of $\Gamma$ for brevity. It is well known that $T$ plays an important role in the study of 2-dimensional (topological) Laguerre planes where ovoidal Laguerre planes are characterized by a 4-dimensional kernel $T$, see [8]. In this note we investigate the kernel $T$ of the automorphism group of a finite Laguerre plane and give a similar characterization of ovoidal planes. More generally, we study such finite Laguerre planes where $T$ is transitive on the set of circles $\mathcal{K}$. 