Spectral Pairs in Cartesian Coordinates

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ABSTRACT. Let $\Omega \subset \mathbb{R}^d$ have finite positive Lebesgue measure, and let $L^2(\Omega)$ be the corresponding Hilbert space of $L^2$-functions on $\Omega$. We shall consider the exponential functions $e_\lambda$ on $\mathbb{R}^d$ given by $e_\lambda(x) = e^{i2\pi \lambda \cdot x}$. If these functions form an orthogonal basis for $L^2(\Omega)$, when $\lambda$ ranges over some subset $\Lambda$ in $\mathbb{R}^d$, then we say that $(\Omega, \Lambda)$ is a spectral pair, and that $\Lambda$ is a spectrum. We conjecture that $(\Omega, \Lambda)$ is a spectral pair if and only if the translates of some set $\Omega'$ by the vectors of $\Lambda$ tile $\mathbb{R}^d$. In the special case of $\Omega = I^d$, the $d$-dimensional unit cube, we prove this conjecture, with $\Omega' = I^d$, for $d \leq 3$, describing all the tilings by $I^d$, and for all $d$ when $\Lambda$ is a discrete periodic set. In an appendix we generalize the notion of spectral pair to measures on a locally compact abelian group and its dual.

1. Introduction

The setting of spectral pairs in $d$ real dimensions involves two subsets $\Omega$ and $\Lambda$ in $\mathbb{R}^d$ such that $\Omega$ has finite and positive $d$-dimensional Lebesgue measure, and $\Lambda$ is an index set for an orthogonal $L^2(\Omega)$-basis $e_\lambda$ of exponentials, i.e.,

$$e_\lambda(x) = e^{i2\pi \lambda \cdot x}, \quad x \in \Omega, \quad \lambda \in \Lambda$$

(1.1)

where $\lambda \cdot x = \sum_{j=1}^d \lambda_j x_j$. We use vector notation $x = (x_1, \cdots, x_d)$, $\lambda = (\lambda_1, \cdots, \lambda_d)$, $x_j, \lambda_j \in \mathbb{R}$, $j = 1, \cdots, d$. The basis property refers to the Hilbert space $L^2(\Omega)$ with inner product

$$\langle f | g \rangle_{\Omega} := \int_{\Omega} f(x) g(x) \, dx$$

(1.2)

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where \( dx = dx_1 \cdots dx_d \), and \( f, g \in L^2(\Omega) \). The corresponding norm is

\[
\| f \|_\Omega := (f, f)_\Omega = \int_\Omega |f(x)|^2 \, dx \, ,
\]

as usual. It follows that the spectral pair property for a pair \((\Omega, \Lambda)\) is equivalent to the nonzero elements of the set

\[
\Lambda - \Lambda = \{ \lambda - \lambda' : \lambda, \lambda' \in \Lambda \}
\]

being contained in the zero-set of the complex valued function

\[
z \mapsto \int_\Omega e^{i2\pi z \cdot x} \, dx =: F_\Omega(z)
\]

and the corresponding \( e_\lambda \)-set \( \{ e_\lambda : \lambda \in \Lambda \} \) being total in \( L^2(\Omega) \). Recall that totality means that the span of the \( e_\lambda \)s is dense in \( L^2(\Omega) \) relative to the \( \| \cdot \|_\Omega \)-norm, or, equivalently, that \( f = 0 \) is the only \( L^2(\Omega) \)-solution to:

\[
(f|e_\lambda) = 0, \quad \text{for all } \lambda \in \Lambda.
\]

Note, \( F_\Omega(z) \) is defined for any \( z = (z_1, \ldots, z_d) \in \mathbb{R}^d \) since \( \Omega \) has finite measure and \( e^{i2\pi z \cdot x} \) has absolute value \( = 1 \). We refer to the book [32] for a summary of the theory of spectral pairs. It was developed in the previous joint papers by the co-authors [13, 14, 15, 16, 17, 18, 19, 20] and elsewhere, e.g., [25, 27, 28, 29]. We recall that Fuglede [6] showed that the disk and the triangle in two dimensions are not spectral sets in the sense that if \( \Omega \) is one of these sets, then there is no possible choice for \( \Lambda \) such that \((\Omega, \Lambda)\) is a spectral pair in \( \mathbb{R}^2 \).

If \( \Omega \subset \mathbb{R}^d \) is open, then we consider the partial derivatives \( \frac{\partial}{\partial x_j}, \, j = 1, \ldots, d \), defined on \( C^\infty_c(\Omega) \) as unbounded skew-symmetric operators in \( L^2(\Omega) \). The corresponding versions \( \frac{1}{i} \frac{\partial}{\partial x_j} \) are symmetric of course. We say that \( \Omega \) has the extension property if there are commuting self-adjoint extension operators \( H_j \), i.e.,

\[
\frac{1}{i} \frac{\partial}{\partial x_j} \subset H_j, \quad j = 1, \ldots, d.
\]

We have (see [6, 11, 15, 30]) the following:

**Theorem 1 (Fuglede, Jorgensen, Pedersen).**

Let \( \Omega \subset \mathbb{R}^d \) be open and connected with finite and positive Lebesgue measure. Then \( \Omega \) has the extension property if and only if it is a spectral set. If \( \Omega \) is only assumed open, then the spectral-set property implies the extension property, but not conversely.

Some of the interest in spectral pairs derives from their connection to tilings. A subset \( \Omega \subset \mathbb{R}^d \) with nonzero measure is said to be a tile if there is a set \( L \subset \mathbb{R}^d \) such that the translates \( \{ \Omega + l : l \in L \} \) cover \( \mathbb{R}^d \) up to measure zero, and if the intersections

\[
(\Omega + l) \cap (\Omega + l') \quad \text{for } l \neq l' \text{ in } L
\]

have measure zero. We will call \((\Omega, L)\) a tiling pair and we will say that \( L \) is a tiling set. The Spectral-Set Conjecture due to Fuglede (see [6, 11, 19, 27, 28, 29, 30]) states:

**Conjecture 1.**

Let \( \Omega \subset \mathbb{R}^d \) have positive and finite Lebesgue measure. Then \( \Omega \) is a spectral set if and only if \( \Omega \) is a tile, i.e., there exists a set \( L \) so that \((\Omega, L)\) is a spectral pair if and only if there exists a set \( L' \) so that \((\Omega, L')\) is a tiling pair.

We formulate a "dual" conjecture.