ABOUT REGIONS OF CONVERGENCE OF EXPANSIONS OF
DIFFERENTIAL EQUATIONS OF THE THREE-DIMENSIONAL
RESTRICTED THREE-BODY PROBLEM IN THE VICINITY OF
COLLINEAR LIBRATION POINTS

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Abstract. The analysis of regions of convergence of expansions of the right-hand sides of the dif-
ferential equations of motion in the vicinity of the collinear libration points in the circular restricted
three-body problem in powers of coordinates of the infinitesimal body due to F. R. Moulton (Moulton,
1920) is shown to be erroneous and his results are corrected.

The generalisation of Moulton’s results to analogous expansions of the equations in the elliptic
problem of three bodies made by R. W. Farquhar (Farquhar, 1968) is shown to be groundless.

Let us consider the motion of an infinitesimal body \( M_2 \) in the gravitational field of
two finite bodies \( M_0 \) and \( M_1 \) revolving around their common barycentre \( G \) on
circular orbits. It is assumed that the infinitesimal body \( M_2 \) does not influence the
motion of the two larger bodies.

Let us introduce Moulton’s synodic barycentric coordinate system (Moulton, 1920):
the \( x \)-axis is oriented positively on the lesser of the primaries \( M_1 \) constantly passing
through \( M_0 \) and \( M_1 \); the \( y \)-axis is oriented perpendicularly to the \( x \)-axis in the direc-
tion of the orbital movement of the primaries; the \( z \)-axis forms a right-handed system
with the axes \( x \) and \( y \).

We shall use also the conventional units of length, mass and time: the constant
distance between \( M_0 \) and \( M_1 \), the sum of the masses of the primaries and the constant
angular rate of the primaries \( M_0 \) and \( M_1 \) around the barycentre \( G \) are taken as units.

If we denote in these units the mass of the lesser of the primaries \( M_1 \) by means of
\( \mu \) then the equations of motion of the third body \( M_2 \) are written as follows:

\[
\begin{align*}
\frac{d^2 x}{dt^2} & - 2 \frac{d y}{dt} = \frac{\partial \Omega}{\partial x} \\
\frac{d^2 y}{dt^2} & + 2 \frac{d x}{dt} = \frac{\partial \Omega}{\partial y} \\
\frac{d^2 z}{dt^2} & = \frac{\partial \Omega}{\partial z}
\end{align*}
\]
where Jacobi’s function $\Omega$ is represented in the form of

$$\Omega = \frac{1}{2} (x^2 + y^2) + \frac{1 - \mu}{r_0} + \frac{\mu}{r_1}$$

and the distances $r_0$ and $r_1$ are determined by the expressions

$$r_0 = \sqrt{(x + \mu)^2 + y^2 + z^2}, \quad r_1 = \sqrt{(x + \mu - 1)^2 + y^2 + z^2}.$$  

It is well known that there are five exact solutions of Equations (1) named ‘points of libration’ and determined from the condition of vanishing of the right-hand sides of these equations:

$$\frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial y} = \frac{\partial \Omega}{\partial z} = 0.$$  

All of these points are located in the orbital plane of the primaries. Two points ($L_4$ and $L_5$) form equilateral triangles with the finite bodies, while the remaining three ($L_1$, $L_2$ and $L_3$) are collinear with them. In references on the restricted three-body problem there is no uniformity in the numbering of the collinear libration points. That is why we shall use Duboshin’s numbering (Duboshin, 1964), i.e. we shall suppose the primaries $M_0$ and $M_1$, their barycentre $G$ and collinear points $L_i (i=1, 2, 3)$ to be placed on the $x$-axis in the order

$$L_1, M_0, G, L_2, M_1, L_3.$$  

For the purpose of studying the motion of the infinitesimal body $M_2$ in the vicinity of the collinear libration points $L_i (i=1, 2, 3)$ the coordinate system $L_i\xi\eta\zeta$ with the origin at the $L_i$-point and axes parallel to the axes of Moulton’s $Gxyz$-system is usually introduced:

$$x = \xi + \alpha_i, \quad y = \eta, \quad z = \zeta,$$

where $\alpha_i$ is the synodic barycentric abscissa of the $L_i$-point.

In new variables $\xi, \eta, \zeta$ Equations (1) have the form:

$$\frac{d^2 \xi}{dt^2} - 2 \frac{d\eta}{dt} = \frac{\partial \Omega}{\partial \xi},$$

$$\frac{d^2 \eta}{dt^2} + 2 \frac{d\xi}{dt} = \frac{\partial \Omega}{\partial \eta},$$

$$\frac{d^2 \zeta}{dt^2} = \frac{\partial \Omega}{\partial \zeta},$$

where

$$\Omega = \frac{1}{2} [(\xi + \alpha_i)^2 + \eta^2] + \frac{1 - \mu}{r_0} + \frac{\mu}{r_1}.$$