GAUGING OF RANDOM FORCES IN A STOCHASTIC DESCRIPTION
OF THERMODYNAMIC SYSTEMS

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The behavior of a system under the influence of random forces which depend on the state of the system is investigated. The Itô-Stratonovich calculus is generalized. A gauging of random forces in the stochastic description of thermodynamic systems is obtained.

1. The behavior of a system that is subject to random influences can be described by a system of stochastic differential equations [1-6]. The group of questions associated with the fluctuation effects of the random forces is traditionally included in the field of statistical physics. Although the reasons for the fluctuations are different (thermal noise, instability, turbulence), the methods of their theoretical description are very similar. One of the reasons for this is that in thermodynamic systems the correlation time of the random disturbances is usually much shorter than the observation time and other characteristic time scales of the system. For such systems the stochastic differential equations [6] lead to a generalized equation of Einstein-Fokker type (this is the reason why this approximation is called the approximation of a random diffusion process).

It is assumed that in all cases the statistical problem (the Einstein-Fokker equation) is completely determined by the system of stochastic equations and the assumptions about the nature of the random forces. However, when random forces depend on the state of the thermodynamic system, such an assumption is not in general true. As a striking example of the nonuniqueness of the transition from the stochastic to the statistical description we
mention the well-known Itô-Stratonovich dilemma [3,5].

In this paper, we study a group of problems associated with the dynamics of statistical systems subject to non-Langevin random forces that depend on the state of the system itself.

2. The state of a statistical system can be described by a set of macroscopic variables \( q \). Their variation with time is due to the unfolding more or less coherently of several microscopic processes.

The microscopic degrees of freedom can be described by introducing random forces in the dynamical equations. At the same time, the macroscopic variables satisfy the following stochastic equations [2,3,4]:

\[
\frac{dq_k(r,t)}{dt} = F_k\{q\} + F'_k\{q,t\},
\]

where \( t \) is the time, and \( F_k \) and \( F'_k \) are, respectively, the deterministic and random forces.

On the other hand, the state of a system that satisfies statistical laws can be described by means of a distribution function \( W(q), W(q,t) \). For the statistical mean of any function of the microscopic state of the system that does not depend explicitly on the time, we have

\[
\langle u \rangle = \int u W dq, \quad \frac{\partial \langle u \rangle}{\partial t} = \int u \frac{\partial W}{\partial t} Dq.
\]

If to the dynamical system (1) we apply the statistical description (2), then the mean denoted in the expression (2) by the angular brackets is the average of \( u \) over the ensemble of realizations of the random forces \( F'_k\{q,t\} \).

Generally speaking, the random forces \( F'_k \) can satisfy a rather complicated distribution law, making it practically impossible to calculate the distribution function \( W \) from the dynamical equations (1). However, as we have already said, for the majority of real physical processes the correlation time of the random forces is short compared with the other characteristic time scales of the system. For processes of such type, we can assume that in a first approximation with respect to the small parameter that is the ratio of the correlation time of the random forces to the other time scales that

\[
\langle F'_k\{q,t\}, F'_m\{q,t'\} \rangle = -2 \Gamma_{km}(q) \delta(t-t').
\]

This equation is satisfied by the following representation of the random forces:

\[
F'_k\{q,t\} = g_{km}(q) \xi^m(t),
\]

where \( g_{km}(q)\xi^m(q) = \Gamma_{mp}(q) \), and the random variable \( \xi^m \) in (4) satisfies a Gaussian random distribution. Here and in all that follows summation over a repeated index is understood.

The system of stochastic equations (1) with the conditions (3) and (4) for the random forces is called the approximation of a random diffusion process. When the amplitudes of the random forces \( g_{km} \) in the expression (4) do not depend on the state of the system \( q \), this approximation leads to a standard equation of Einstein-Fokker type.

To describe a system with random forces that depend on the macroscopic variables \( q \), we construct a difference scheme, dividing the time into equal intervals \( \delta t \) by a discrete sequence of points \( t_i \). Then the effect of the deterministic and random forces can be described by the difference equations

\[
\delta q_k(t_i) = F_k\{q\}\delta t + g_{km}(q)\delta \xi^m(t_i).
\]

We have here introduced the variations of the macroscopic, \( q \), and microscopic, \( \xi^k \), variables in step 1:

\[
\delta q_k(t_i) = q_k(t_i) - q_k(t_{i-1}), \quad \delta \xi^k(t_i) = \xi^k(t_i) - \xi^k(t_{i-1}).
\]

The condition that \( \xi \) satisfy a random Gaussian distribution can be represented accordingly in the form

\[
\langle \delta \xi^m(t_i) \rangle = 0, \quad \langle \delta \xi^m(t_i) \delta \xi^m(t_j) \rangle = 2\delta t \delta_{ij} \delta_{km}.
\]

In the expression (5) the macroscopic variables \( q_k \) of the state of the thermodynamic system appear as arguments of both the deterministic and the random forces. The question