Long-time behaviour for weakly damped driven nonlinear Schrödinger equations in $\mathbb{R}^N$, $N \leq 3$

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Abstract
We study the long-time behaviour of solutions to nonlinear Schrödinger equations with a zero order dissipation and an additional external force, when the space variable $x$ varies over $\mathbb{R}^N$, $N \leq 3$. We prove that the long-time behaviour is described by a maximal compact attractor for the strong topology of $H^1(\mathbb{R}^N)$.

1 Introduction
In this work, we investigate the long-time behaviour of the solutions to weakly damped driven nonlinear Schrödinger equations in $\mathbb{R}^N$, $N \leq 3$,

$$
\begin{align*}
iu_t + \Delta u + g(|u|^2) u + i\gamma u & = f \quad \text{in } \mathbb{R}^N \times (0, +\infty), \quad (1.1) \\
u(0) & = u_0 \quad \text{in } \mathbb{R}^N. \quad (1.2)
\end{align*}
$$

Such an equation arises for instance in plasma physics ([12]) or in optical fibers models ([4]).

The long-time behaviour of the solutions to (1.1) differs broadly between the case $\gamma = 0$, $f = 0$ and the other cases; in the latter, the trajectories of the solutions to (1.1) get trapped in some bounded subset of the phase space as $t \to +\infty$, because the zero order dissipation ($\gamma > 0$) and the external force $f$ are taken into account. More precisely, we shall prove in that case that, under suitable assumptions on the nonlinearity $g$, the long-time behaviour of the solutions to (1.1) – (1.2) is described by a maximal compact attractor in $H^1(\mathbb{R}^N)$. This question has already been studied by J.M. Ghidaglia, when the space variable $x$ varies over a bounded interval of $\mathbb{R}$ and for various boundary conditions ([8]). In this case, J.M. Ghidaglia has proved the existence of the maximal attractor for the weak topologies of $H^1$ and $H^2$, and has provided estimates of the fractal dimension in $H^1$ of these attractors. More recently, the existence of a maximal compact attractor in $H^1$ for (1.1) has
been addressed by M. Abounouh, when $x$ varies over an open bounded subset of $\mathbb{R}^2$ ([1]).

Hereafter, we intend to study the existence of the maximal compact attractor in $H^1(\mathbb{R}^N)$ for (1.1). As usual when one deals with existence of compact attractors for evolution equations on unbounded domains, an additional difficulty arises from the lack of compactness of the embedding of $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$.

However, a splitting method in the spirit of [10] enables us to get the compactness of the trajectories of (1.1) in $L^2(\mathbb{R}^N)$. Similar splitting methods have already been used by E. Feireisl for semilinear damped wave equations in $\mathbb{R}^m$ ([6]) and by E. Feireisl and coworkers for reaction-diffusion equations in $\mathbb{R}^m$ ([7]). Finally, a technique first used by J. Ball ([3]) and further developed in [9] and in [1] yields afterwards the compactness of the trajectories of (1.1) in $H^1(\mathbb{R}^N)$.

We now state the assumptions on $\gamma$, $g$ and $f$:

\begin{itemize}
  \item [(H1)] $\gamma > 0$,
  \item [(H2)] $g \in C^1([0, +\infty), \mathbb{R})$ and there exist $\alpha_1 > 0$, $\alpha_2 > 0$ and $\delta \in \left[0, \frac{2}{N}\right)$ such that
    \begin{align*}
      G(r) &\leq \alpha_1 r \left(1 + r^\delta\right), \quad r \geq 0, \\
      r g(r) - G(r) &\leq \alpha_2 r \left(1 + r^\delta\right), \quad r \geq 0,
    \end{align*}
    \end{itemize}

where

$$G(r) = \int_0^r g(s) \, ds, \quad r \geq 0.$$  

Moreover, when $N = 2$ or 3, we assume that there exists $\alpha_3 > 0$ such that, for $(\xi, \xi') \in C^2$,

$$|g(|\xi|^2) \xi - g(|\xi'|^2) \xi'| \leq \alpha_3 \left(1 + |\xi|^\alpha + |\xi'|^\alpha\right) |\xi - \xi'|,$$

where $\alpha \in [0, 2^* - 2)$ (where $2^* = +\infty$ if $N = 2$ and $2^* = 6$ if $N = 3$).

\begin{itemize}
  \item [(H3)] $f \in L^2(\mathbb{R}^N)$.
\end{itemize}

Note that the nonlinearity $|u|^{2\sigma} u$, $0 \leq \sigma < \frac{2}{N}$ ($g(r) = r^\sigma$) satisfies (H2), as well as $-|u|^{2\sigma} u$, $0 \leq 2\sigma < 2^* - 2$.

Under the above assumptions, we have the following existence result:

**Proposition 1.1** Let $u_0 \in H^1(\mathbb{R}^N)$. Under assumptions (H1) -- (H3), the problem (1.1) -- (1.2) has a unique solution

$$u \in C([0, +\infty), H^1(\mathbb{R}^N)) \cap C^1([0, +\infty), H^{-1}(\mathbb{R}^N)),$$

and the mapping $S_t : u_0 \mapsto u(t)$ is continuous in $H^1(\mathbb{R}^N)$. 