ISOMETRIC IMMERSIONS OF RIEMANNIAN SPACES
IN EUCLIDEAN SPACES

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Questions of the theory of isometric immersions of Riemannian spaces in Euclidean spaces beginning with the very first results on this topic and also results on immersions of pseudo-Riemannian spaces in pseudo-Euclidean spaces and applications of the theory of immersions in the general theory of relativity are considered.

The paper is devoted to a survey of works on isometric immersions of Riemannian and pseudo-Riemannian spaces in Euclidean and pseudo-Euclidean spaces.

The question of immersions of Riemannian spaces is connected with two distinct approaches to the problem of studying Riemannian manifolds. The first of these consists in investigating an abstractly defined manifold. The second consists in investigating a Riemannian manifold as a submanifold of Euclidean space. The following question arises naturally: Is every n-dimensional Riemannian manifold \( V^n \) a submanifold of Euclidean space \( E^N \)? In the most general formulation this question was solved positively in the fifties and sixties of our century by the American mathematician Nash [141, 142, 143]. The investigations of Kuiper [122, 123] are closely related to those of Nash. Although the results of Nash and Kuiper are of universal character, they cannot be considered definitive, since they do not give a complete answer to the very important question of the choice of the optimal dimension \( N \) of the Euclidean space \( E^N \) in which a given \( V^n \) or some class of Riemannian spaces is immersed. The corresponding problematics will be formulated in the paper, and a survey of results will be given.

In the paper major coverage is given to papers on immersions of pseudo-Riemannian spaces in pseudo-Euclidean spaces. Interest in this theme is to considerable extent connected with various problems of theoretical physics and theoretical astronomy. Clarifications of the physical character of corresponding results will be given along with a survey of papers on this topic.

We shall not consider in detail the results of immersions of two-dimensional Riemannian metrics, since the surveys [21, 22] are devoted to this problem, while the fundamental papers of Aleksandrov [1] and Pogorelova [18] deal with immersions of two-dimensional metrics of positive curvature. We shall consider these questions and also questions related to immersions in curved spaces only to the extent that they aid in understanding the history of the development of the problem considered.

The bibliography extends to the end of 1976. We shall use the following notation: \( E^n \) is n-dimensional Euclidean space, \( F^n(p,q) \) is n-dimensional pseudo-Euclidean space with signature (\( p, q \)).

1. A Survey of Papers on Isometric Immersions up to 1950

1. Formulation of the Problem. Basic Results. The problem of isometrically embedding of Riemannian space \( V^n \) in some Euclidean space \( E^N \) was first formulated by Schlaefli [167] in 1873 and, so it seemed to him, not only formulated but also solved. Schlaefli obtained the following equations (we shall henceforth call them the Schlaefli equations):

\[r_i r_j = g_{ij},\]  \hspace{1cm} (1)

in which \(g_{ij} = g_{ij}(x^1, \ldots, x^n)\) are the coordinates of the metric tensor of the Riemannian space \(\mathbb{V}^n, x^1, \ldots, x^n\), intrinsic coordinates; and \(r_i = \partial r / \partial x^i\), where \(r = r(x^1, \ldots, x^n)\), is the desired radius vector of the submanifold \(M^n\) in \(E^N\) on which an inner metric is induced which coincides with the metric of \(\mathbb{V}^n\).

Since the number of equations (1) is equal to \(s_n = n(n + 1)/2\), Schlaefli was convinced that at least locally* \(\mathbb{V}^n\) can be embedded in \(E^{s_n}\). In other words, the question of the existence of solutions of the system (1) for \(N = s_n\) did not arise [in this case the number of equations and the number of unknown functions – the coordinates of the vector \(r(x^1, \ldots, x^n)\) – coincide].

For analytic Riemannian metrics \(\mathbb{V}^n\) the local Schlaefli problem was solved in the work of Janet [114] (1926), Cartan [74] (1927), and Burstin [73] (1931).

To solve the problem of the embedding of analytic Riemannian manifolds \(\mathbb{V}^n\) in \(E^{s_n}\) Cartan made use of the tools of the method of outer forms which he created. It should be noted that so far Cartan’s proof has not been simplified nor have the basic ideas of his arguments been clarified.

In the work cited of Janet for the linear element† \(ds^2 = g_{11}(dx^1)^2 + g_{ij}dx^idx^j\) (i > 1, j > 1) the Schlaefli equations [see (1)] were brought to the form

\[r_i r_k = 0, \quad k = 1, 2, \ldots, n,\]

\[r_{11} r_{1m} = r_{11} r_{1m} = -\frac{1}{2} \frac{\partial^2 g_{i1m}}{\partial x^i \partial x^m}, \quad l > 1, \quad m > 1\]  \hspace{1cm} (2)

by differentiations of the simplest algebraic operations. It is clear that if all the vectors \(r_k, r_{lm}\) [the number of these vectors is equal to \(n(n + 1)/2\)] are linearly independent, then the system (2) can be solved for the coordinates of the vector \(r^1\). As a result, a system of Cauchy–Kovalevskaya type is obtained for which a solution exists. Janet did not prove the possibility of a choice of initial data for which the linear independence of the vectors \(r_k, r_{lm}\) is ensured. This plan was realized by Burstin. He constructed an inductive process for the isometric embedding in \(E^{s_n}\) of specially chosen submanifolds in \(\mathbb{V}^n\) of increasing dimensions such that at the \(n\)-th step of this process the desired isometric immersion of \(\mathbb{V}^n\) in \(E^{s_n}\) is obtained, while for this immersion all the vectors \(r_k, r_{lm}\) are linearly independent. This type of isometric immersion of \(\mathbb{V}^n\) in \(E^N\) subsequently became known as a free immersion.

2. Problem of the Class of a Riemannian Metric. The Problem of Nonimmersibility. In 1886 Schur published the work [169] in which the possibility is established of the local, analytic, isometric immersion of Lobachevskii space \(\mathbb{H}^n\) in \(E^N\) for \(N = 2n - 1\). This result is very important in clarifying the problems of the theory of isometric immersions.

The relation of dimensions (\(n\) and \(N = 2n - 1\)) of the immersed space \(\mathbb{H}^n\) in the Euclidean space \(E^N\) in Schur’s result differs sharply from the relation of the dimensions (\(n\) and \(N = s_n = n(n + 1)/2\)) in the general result (Schlaefli, Janet, Cartan, and Burstin). It is therefore natural to formulate the following two important problems of the theory of isometric immersions – the problem of the class of the Riemannian metric and the problem of nonimmersibility.

The problem of the class of the Riemannian metric of \(\mathbb{V}^n\) consists in resolving the question of the minimal dimension \(N\) of the Euclidean space \(E^N\) in which \(\mathbb{V}^n\) can be isometrically immersed. The difference \(N - n\) is called the class of the Riemannian metric of \(\mathbb{V}^n\).

In its original formulation this question pertained to analytic immersions of analytic Riemannian metrics. It became clear only in the fifties that the differentiability conditions in this problem are very basic; it followed from the remarkable result of Nach [141] that if the immersion is only required to be of class \(C^1\), then locally the class of all Riemannian metrics is equal to 1, i.e., locally all Riemannian metrics of dimension \(n\) can be immersed as hypersurfaces of class \(C^1\) in \(E^{n+1}\).

We shall subsequently discuss various aspects of this problem of the metric class in surveying other works on the theory of immersions.

The second important problem – the problem of the immersibility of a given Riemannian \(n\)-dimensional manifold in Euclidean space \(E^N\) of given dimension \(N\) – was first formulated by Hilbert in 1900 in his famous Problems [104]. In these Problems Hilbert posed the problem of the existence of \(E^3\) of a complete surface of

*At the time of Schlaefli’s work local and global embedding of a manifold were not distinguished.
†Locally the linear element of \(\mathbb{V}^n\) can always be brought to the indicated form.