On Lower Bounds of Exponential Frames

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Communicated by Ingrid Daubechies

ABSTRACT. Lower frame bounds for sequences of exponentials are obtained in a special version of Avdonin's theorem on "1/4 in the mean" [1] and in a theorem of Duffin and Schaeffer [4].

1. Introduction

The notion of frame has been introduced by Duffin and Schaeffer [4]. A sequence \((\varphi_n)_{n \in \mathbb{Z}}\) in a Hilbert space \((H, \langle \cdot, \cdot \rangle_H)\) is a frame for \(H\), if there exist positive constants \(A, B\) such that for all \(f \in H\):

\[
A \|f\|_H^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, \varphi_n \rangle_H|^2 \leq B \|f\|_H^2.
\]

More specifically, \((\varphi_n)_{n \in \mathbb{Z}}\) is called an \((A, B)\)-frame. The constants \(A\) and \(B\) are called lower and upper frame bounds, respectively. A frame is exact if it is no longer a frame after any of its elements are removed.

Duffin and Schaeffer [4] have given a sufficient condition for a sequence of exponentials \((e^{i\lambda_n})_{n \in \mathbb{Z}}\) to be a frame for \(L^2(-\gamma, \gamma), \gamma > 0\). Avdonin [1] has given a sufficient condition for a sequence of exponentials \((e^{i\lambda_n})_{n \in \mathbb{Z}}\) to be an exact frame for \(L^2(-\pi, \pi)\). In both papers, only the mere existence of a lower bound is proved.

Kölzow [7] asked for explicit lower frame bounds, in terms of the data by which the sequence \((\lambda_n)_{n \in \mathbb{Z}}\) is restricted. In this paper, we shall obtain a lower bound in a special version of Avdonin's theorem on "1/4 in the mean" (Theorem 1). The result will be used to obtain a lower bound in a theorem of Duffin and Schaeffer (Theorem 2). Finally, an application to irregular sampling is pointed out.

Math Subject Classifications. 42C15, 30A10, 94A12.
Keywords and Phrases. lower bound, exponential frame, sine-type-function, irregular sampling.
Acknowledgements and Notes. The results of this paper are part of the author's forthcoming doctoral thesis and were presented, in a preliminary form, at the 1997 International Workshop on Sampling Theory and Applications, June 16-19, 1997, Aveiro, Portugal.
2. Preliminary Remarks and Main Results

$PW^2_\sigma$ denotes the Paley–Wiener space of entire functions of exponential type at most $\sigma$, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. For a sequence $(\lambda_n)_{n \in \mathbb{Z}}$ of real numbers the classical Paley–Wiener theorem yields that $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ is an $(A, B)$-frame for $L^2(-\sigma, \sigma)$ if and only if

$$A \| F \|_{PW^2_\sigma}^2 \leq 2\pi \sum_{n \in \mathbb{Z}} |F(\lambda_n)|^2 \leq B \| F \|_{PW^2_\sigma}^2 \quad \forall F \in PW^2_\sigma.$$ 

A sequence $(\lambda_n)_{n \in \mathbb{Z}}$ of complex numbers is called separated by $\delta > 0$, if

$$|\lambda_n - \lambda_m| \geq \delta \quad \forall m, n \in \mathbb{Z} : m \neq n.$$ 

The following lemma asserts an upper bound for separated sequences with bounded imaginary parts. The proof follows the same argument as in the proof of Lemma 2 in Katsnel'son [5], using $\text{card}(n \in \mathbb{Z} : |z - \lambda_n| \leq 1) \leq (1 + 2/\delta)^2 \forall z \in \mathbb{C}$.

**Lemma 1.**

Let $(\lambda_n)_{n \in \mathbb{Z}}$ be a sequence of complex numbers, separated by $\delta > 0$, with imaginary parts bounded by $\tau < \infty$. Then,

$$\sum_{n \in \mathbb{Z}} |F(\lambda_n)|^2 \leq \frac{2}{\pi} \left( \frac{2}{\delta} + 1 \right)^2 \cdot \frac{e^{2\sigma(\delta + 1)} - 1}{2\sigma} \| F \|_{PW^2_\sigma}^2 \quad \forall F \in PW^2_\sigma$$

for any $\sigma > 0$.

The main results of this paper are the following:

**Theorem 1 (A lower bound in the Theorem of Avdonin).** Let $(\delta_k)_{k \in \mathbb{Z}}$ be a sequence of real numbers, bounded by $L < \infty$. Suppose $(k + \delta_k)_{k \in \mathbb{Z}}$ is separated by $\delta > 0$, and there are $d \in [0, 1/4)$ and a natural number $N$, such that

$$\sum_{k = jN + 1}^{(j+1)N} \delta_k \leq N \cdot d \quad \forall j \in \mathbb{Z}.$$ 

Then $(e^{i(k+\delta_k) \cdot})_{k \in \mathbb{Z}}$ is an exact frame for $L^2(-\pi, \pi)$, and the following constant is a lower frame bound:

$$A_{AV}(L, \delta, d, N) := e^{-20\pi^2(2L)^2N^2/N^2} \cdot \frac{240(2L)^2}{\delta \pi (1/4 - d)} \cdot \left( \frac{\bar{\delta}}{9L} \right)^{2N},$$

where

$$\bar{N} := N \cdot \left[ \frac{1}{N} \cdot \frac{2(4L + 2)^2}{1/4 - d} \right], \quad \bar{\delta} := \frac{3}{2} + 2(3L + 1), \quad \bar{L} := \frac{1}{2} \left( \frac{1}{4} - d \right) \delta.$$ 

$\square$

($[x]$ denotes the smallest integer greater than or equal to $x$.)

**Theorem 2 (A lower bound in the Theorem of Duffin and Schaeffer).** Suppose $(\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of real numbers, separated by $\delta > 0$. Let $\sigma > 0$, $L \geq 0$, $0 < \gamma < \pi \sigma$ and suppose $(\lambda_n - n/\sigma)_{n \in \mathbb{Z}}$ is bounded by $L$. Then $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ is a frame for $L^2(-\gamma, \gamma)$, and the following constant is a lower frame bound:

$$A_{DS}(L, \delta, \sigma, \gamma) := \frac{\gamma}{\pi} \cdot e^{-5\pi^2 \left( \frac{12M + 1}{M^2(M + 1)^2} \right)^{4M(M+1)}} \cdot \left( \frac{\delta \gamma / \pi}{308M} \right)^{240(12M + 1)^{2M(M+1)}}.$$