**NUMERICAL-ANALYTICAL METHOD OF ANALYSIS OF THE NORMAL MODES OF A VERTICAL COLUMN**

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A numerical-analytical method is described for analyzing the effect of weight on the normal modes in a vertical column. Convergence of the method is considered and numerical results are reported.

One of the main problems in the mechanics of one-dimensional structures (rods, molds, arches) is the analysis of free oscillations and the closely related problems of static stability. The corresponding boundary-value problems are usually solved numerically by finite-element or finite-difference methods. These universal methods are independent of the structure of the coefficients of the differential equations of motion and produce the frequencies and the forms of the normal modes with an accuracy which is sufficient for practical applications. However, like all numerical methods, they require a collection of test or control problems that are solvable with high accuracy by independent analytical or numerical-analytical methods.

The applicability of numerical-analytical methods depends on the structure of the equations of motion. The power series method has a fairly wide scope. It reduces the differential eigenvalue problem to an algebraic eigenvalue problem for a one-dimensional recurrence formula. This problem in turn is amenable to certain algebraic transformations utilizing the polynomial dependence of the coefficients of the solution power series on the eigenvalues. These transformations of the frequency produce the roots of some polynomial with explicit expressions for the coefficients. On the whole, this is a high-accuracy method, i.e., the method error ("truncation" error) can be made as small as desired, specifically less than the machine rounding error.

As a reference problem in the theory of oscillation of rod structures we consider the problem of free oscillations of a homogeneous vertical column compressed longitudinally by its own weight (the bottom end of the column is fixed and the top end is free). This problem is not new, nor is the solution method new; however, we stress some features of the application of the power series method, which convert it into a "high-accuracy" method in the above sense. The problem is attractive because it has analytical solutions in two limiting cases: with zero gravity (natural modes of a homogeneous console) and zero frequency (the Euler problem of static stability of a heavy column). These analytical solutions can be used to monitor the accuracy of the numerical-analytical method.

Of main interest is the construction of the "eigenlines" of the problem, i.e., finding pairs of parameter values "frequency-weight" for which the problem has nontrivial solutions.

The equation of small oscillations of a longitudinally compressed rod in dimensionless variables has the form [1]

$$W''(x) + \mu((1 - x)W'(x))' - \Omega^2 W(x) = 0 \quad (1)$$

with the boundary conditions

$$W(0) = W'(0) = 0; \quad W''(1) = W'''(1) = 0, \quad (2)$$

where $x$ is the dimensionless coordinate along the rod; $\mu = \beta \rho g S/EI$; $\Omega^2 = l^2 \rho So^2/EI$; $l$ is the length of the rod; $\rho$ is the rod density; $S$ is the rod cross section; $g$ is the free fall acceleration; $E$ is Young's modulus; $I$ is the moment of inertia; $\omega$ is the eigenfrequency of the rod; $W(x)$ is the deflection function of the rod.
The dimensionless parameters $\mu$ and $\Omega$ represent the ratio of the weight and the eigenfrequency of the rod to its elastic characteristics. The structure of the coefficients in Eq. (1) makes it possible to find its solution in series form in powers of the coordinate $x$:

$$W(x) = \sum_{n=2}^{\infty} \frac{a_n}{n!} x^n.$$  \hspace{1cm} (3)

The sum starts with $n = 2$ in order to automatically satisfy the boundary conditions at the bottom end of the rod. Substituting (3) in (1), we obtain a five-point recurrence for the coefficients $a_i$:

$$a_{i+4} = -\mu a_{i+2} + \mu (i + 1)a_{i+1} + \Omega^2 a_i, \quad i = 0, \infty$$  \hspace{1cm} (4)

with the initial conditions $a_0 = a_1 = 0$. Since (4) is a linear formula, the coefficients $a_i$ are representable as a linear combination of $a_2$ and $a_3$:

$$a_i = a_3 A_i + a_2 B_i,$$

where the coefficients $A_i$ and $B_i$ are also determined by the recurrence (4) with the initial conditions

$$A_0 = A_1 = A_2 = 0, \quad A_3 = 1; \quad B_0 = B_1 = B_3 = 0, \quad B_2 = 1.$$

Substituting these coefficients $a_i$ in (3), we write the solution $W(x)$ as the sum of two fundamental solutions

$$W(x) = a_3 W_2(x) + a_3 W_3(x),$$

where

$$W_2(x) = \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n, \quad W_3(x) = \sum_{n=2}^{\infty} \frac{A_n}{n!} x^n.$$

Several first coefficients $A_i$ are easily seen to be polynomials in the parameters $\mu$ and $\Omega^2$ the degree of which is independent of the index of the coefficient. Treating $A_i$ as polynomials in powers of $\mu$ with coefficients that depend on $\Omega^2$, we can establish the following relationship:

$$A_i = \sum_{k=0}^{f_1(i)} \alpha_{i,k+1} \mu^k,$$

where $f_1(\Omega) = \left[\frac{i-3}{2}\right]$ and $\alpha_{i,k+1} = \alpha_{i,k+1}(\Omega^2)$. Square brackets denote the whole part of the number.

The coefficients $A_i$ also can be treated as polynomials in powers of $\Omega$ with coefficients that depend on $\mu$. Substituting the polynomial representations of the coefficients in formula (4) and separating between even and odd indices, we obtain two-index matrix recurrences for $\alpha_{i,k}$.

For odd indices $i$, $i = 2n - 1$, $n = 1, \ldots, \infty$:

for $k = 0$ \hspace{1cm} $\alpha_{2n+3,1} = \Omega^2 \alpha_{2n-1,1}$ ;
for $k = 1, n-2$ \hspace{1cm} $\alpha_{2n+3,k+1} = -\alpha_{2n+1,k} + 2n \alpha_{2n,k} + \Omega^2 \alpha_{2n-1,k+1}$ ;
for $k = n-1$ \hspace{1cm} $\alpha_{2n+3,n} = -\alpha_{2n+1,n-1} + 2n \alpha_{2n,n-1}$ ;
for $k = n$ \hspace{1cm} $\alpha_{2n+3,n+1} = -\alpha_{2n+1,n}$ .

For even $i$, $i = 2n$, $n = 1, \ldots, \infty$:

for $k = 0$ \hspace{1cm} $\alpha_{2n+4,1} = \Omega^2 \alpha_{2n,1}$ ;
for $k = 1, n-2$ \hspace{1cm} $\alpha_{2n+4,k+1} = \alpha_{2n+2,k} + (2n + 1) \alpha_{2n+1,k} + \Omega^2 \alpha_{2n,k+1}$ ;
for $k = n-1, n$ \hspace{1cm} $\alpha_{2n+4,k+1} = \alpha_{2n+2,k} + (2n + 1) \alpha_{2n+1,k}$ .

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