PROPAGATION OF SHEAR WAVES IN A SEMI-INFINITE ELASTIC WAVEGUIDE WITH A LOCAL CONSTRICION

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A procedure is proposed for computing longitudinal-shear stresses and displacements in a semi-infinite waveguide with two lateral cutouts. Numerical results are reported for the stresses on the cutout contours and for the transmission coefficient of the elastic wave.

Plane waveguides allow various deviations of the shape of the propagation region from the canonical strip shape. Waveguides may have local constrictions, curved sections, and corrugated walls [2].

In this paper, we analyze the wave field of longitudinal-shear displacements and stresses in a semi-infinite elastic waveguide with a local constriction. The end section of the waveguide has an arbitrary contour. We use the method of solution of dynamic problems of elasticity theory for bodies with curvilinear cuts proposed in [6] and generalized to periodic problems in [3].

Assume that the waveguide is an elastic body cylindrical in the direction of the Oz axis and bounded in plan by the straight lines \(x = 0, x = a\) and the contours \(L_1, L_2, L_3\) (Fig. 1). A time-harmonic load \(X_n = Y_n = 0, Z_n = \text{Re}\{f(s)e^{-\omega t}\}\) is applied to the edge \(L_3\) (\(s, n\) are the circular coordinate and the normal to the contour, \(\omega\) is the circular velocity, \(t\) is time). The harmonic load excites a longitudinal shear wave that propagates along the waveguide.

Diffraction of the wave on the local constriction represented by the two lateral cutouts \(L_2, L_3\) creates a zone of increased stresses. The constriction in turn perturbs the amplitude of the elastic wave traveling along the waveguide.

The antiplanar strain is uniquely determined by the only nonzero displacement \(w(x, y) = \text{Re}\{W(x, y)e^{-\omega t}\}\) along the Oz axis, whose amplitude \(W(x, y)\) satisfies the Helmholtz equation

\[\Delta W + \gamma^2 W = 0 \quad (\gamma = \sqrt{\rho/\mu \omega}),\]

where \(\Delta\) is the differential Laplace operator, \(\gamma\) is the wavenumber, \(\rho\) is the density of the medium, \(\mu\) is the shear modulus. The longitudinal-shear stresses are expressed in terms of the displacement \(w\) in the form

\[\tau_{xz} = \mu \partial w / \partial x, \quad \tau_{yz} = \mu \partial w / \partial y.\]

Assume that the straight edges \(x = 0, x = a\) of the waveguide are fixed, i.e.,

\[W(0, y) = W(a, y) = 0.\]

The boundary conditions at the end \(L_1\) and in the cutouts \(L_2, L_3\) are represented in the form

\[\mu \partial W / \partial n_{L_1} = f(s), \quad \partial W / \partial n_{L_2} = \partial W / \partial n_{L_3} = 0.\]

The integral representation of the solution of the boundary-value problem is constructed using Green’s formula for a function that satisfies the Helmholtz equation [4]. We obtain
Here $L = \cup L_m$ is the curvilinear boundary of the waveguide region $D$; $p(s) = p_m(s) = W \big|_{L_m}$, $s \in L_m$; $p_m(s)$ are the unknown densities ($m = 1, 2, 3$); $G$ is Green's function of the boundary-value problem (1), (3) for the strip $0 \leq x \leq a$, $-\infty < y < \infty$.

$$W = \int_L p(s) \frac{\partial G}{\partial n} ds - \frac{1}{\mu} \int f(s) G ds.$$  \hfill (5)

It follows from (6) that the solution (5) for $y \to \infty$ is represented as a superposition of traveling and exponentially decaying waves that propagate in a regular waveguide. The integral representation (5) thus satisfies the partial radiation conditions for wave-type regions [5]. If $\eta = y$, the series in (6) is weakly convergent, and for $\xi = z$ it diverges. To improve the convergence of the series, we isolate its principal part that corresponds to Green's function of the static problem. We obtain

$$G = G_0 + G_1,$$

$$G_0 = G_{y = 0} = \frac{1}{2\pi} \ln \left| \frac{\sin \frac{\pi}{2a}(\xi - z)}{\sin \frac{\pi}{2a}(\xi + z)} \right|;$$

$$G_1 = -\frac{1}{2\pi} \sum_k \left( \frac{1}{\lambda_k} e^{-\lambda_k \eta - y} - \frac{1}{\lambda_k} e^{-\lambda_k \eta + y} \right) \sin \alpha_k \xi \sin \alpha_k x.$$  \hfill (7)

Hence we see that the function $G$ (7) satisfies Eq. (1) when $\xi \neq z$ and has a logarithmic singularity when $r = |z - \xi| \to 0$. The general term of the series for $G_1$ decays as $k^{-3}$ for $r = 0$ and exponentially for $\eta \neq y$.

The normal derivative of the function $W$ (5) is integrated by parts using the identity

$$\frac{\partial^2 G}{\partial n \partial n_0} = -\frac{\partial^2 G}{\partial s \partial s_0} + 2e^{(\rho - \varphi_0)} \frac{\partial^2 G}{\partial \xi \partial s} + 2e^{(\rho - \varphi_0)} \frac{\partial^2 G}{\partial \xi \partial s_0},$$  \hfill (8)

and we pass to the limit as $z \to z_0 = \xi_0 + i\eta_0 \in L$ using the standard theorems of potential theory [4]. Substituting the limit values of $\delta W/\delta n$ in the boundary conditions (4), we obtain a system of three singular integrodifferential equations for the complex densities $p_m(s)$, $m = 1, 2, 3$, which for simplicity is rewritten as one integral equation over the contour $L$ with density $p(s)$:

$$\int_L \left( p'(s) \frac{\partial G}{\partial s_0} + p(s)M(s, s_0) \right) ds = \frac{1}{2\mu} \int f(s_0) ds + \frac{1}{\mu L_1} \int f(s) \frac{\partial G}{\partial s_0} ds,$$

$$M(s, s_0) = 2e^{(\rho - \varphi_0)} \frac{\partial^2 G}{\partial \xi \partial s_0}, \quad e^\rho = \frac{\partial \rho}{ds}, \quad e^{\rho_0} = \frac{\partial \rho_0}{ds_0}.$$  \hfill (9)

Here the kernel $\delta G/\delta s_0$ is singular; $M(s, s_0)$ is regular; $f(s_0) = 0$ if $s_0 \not\in L_1$.

To ensure unique solvability of the system of integrodifferential equations, we need the additional conditions

$$\int_{L_m} p'(s) ds = 0, \quad m = 1, 2, 3,$$  \hfill (10)

which ensure zero displacement at the end points of the contours $L_m$ that merge into the fixed straight edges $x = 0, x = a$ of the waveguide.

Each of the three contours is representable in parametric form as $s = \tilde{s}(\beta)$, $s_0 = \tilde{s}(\beta_0)$, $-1 \leq \beta, \beta_0 \leq 1$. The solution of the system of integral equations (9) is sought in the form

$$p'(s) = \left( \sqrt{1 - \beta^2} \Omega_m(\beta) + \frac{1 - \beta}{2} A_m + \frac{1 + \beta}{2} B_m \right) s'(\beta), \quad m = 1, 2, 3.$$  \hfill (11)

Here $\Omega_m(\beta)$ are regular functions, $A_m, B_m$ are unknown constants.