AUTOMORPHISMS OF GRAPHS

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We consider the description of automorphisms of the Berge graph in terms of the graph mapping. Necessary and sufficient conditions for a permutation to be an automorphism are given.

Let \( G = (X, F) \) be a Berge graph [1] in which \( X \) is the vertex set and \( F: X \to X \) is a mapping that associates to each \( x \in X \) the set of vertices reached by arcs from \( x \). Moreover, let \( \rho \) be a partition of the set \( X \) into pairwise nonintersecting classes and \( X/\rho \) the set of classes generated by the partition.

**Definition.** An automorphism of the graph \( G = (X, F) \) is a bijection \( \varphi: X \to X \) such that for all \( x \in X \) and \( y \in X \) that satisfy the relationship \( \varphi(x) = y \) we have the equality

\[
\varphi(Fx) = F(\varphi(x)).
\]

The automorphism group of the graph \( G \) can be computed given the properties of automorphism cycles of the graph relative to the graph mapping. Each computed automorphism induces a cyclic subgroup of the automorphism group of the graph and by the finiteness of the graph the entire graph group [3] is exhausted by a finite number of such cyclic subgroups [2].

Let \( \rho \) be a partition of the set \( X \) into pairwise nonintersecting classes \( \sigma_k \) such that if we denote by \( \tilde{F} \) the extension of \( F \) to the set of classes \( X/\rho \), then \( \tilde{F}\sigma_k \) is a set of classes from \( X/\rho \).

Fix some order on each class from \( X/\rho \). Order the elements from \( \tilde{F}\sigma_k \) for each \( x \in X \) so that all the elements that have the same index after ordering are contained in the same class from \( X/\rho \).

The set of elements from \( \tilde{F}\sigma_k \) that have the same index \( m \) after ordering is denoted \( \tilde{F}(m)\sigma_k \) or \( F(m)\sigma_k \). By \( p \sigma \) we denote the class \( \sigma \) repeated \( p \) times with some fixed order; the relationship \( (F\sigma_k)_m \approx p\sigma(m); \sigma_k, \sigma \in X/\rho; p \geq 0 \) implies that the right-hand side is a collection of \( p \) copies of the set \( \sigma \) ordered in the same way, and the order on \( \sigma \) is identical with the original order on this class up to a cyclic shift.

**THEOREM.** The partition \( \rho \) of the set \( X \) of the graph \( G = (X, F) \) induces an automorphism \( g \) if and only if for any class \( \sigma_k \in \rho \) and any \( m > 0 \) there exists \( \sigma_{k(m)} \in \rho \) such that

\[
(F\sigma_k)_m \approx p\sigma_{k(m)} \quad \text{for} \quad |\sigma_k| > 1;
\]

\[
F\sigma_k = \{\sigma_k\}; \quad \text{for} \quad |\sigma_k| = 1.
\]

**Proof.** Necessity. Let \( \rho \) be a partition of the set \( X \) into classes \( \{\sigma_k\} \) which are ordered cyclically and induce the automorphism \( g \), and \( x \in \sigma_k \), \( y \in \sigma_k \): \( |\sigma_k| \leq |\sigma_k| \leq |\sigma_k| > 1 \) such that \( F(m)x = y \).

For an arbitrary but fixed \( m > 0 \), partition the set \( \sigma_k \) into classes \( \sigma_{k1} \), assigning to the same class the elements \( z \in \sigma_{k1} \) and \( z' \in \sigma_{k1} \) that satisfy the relationships \( F(m)z = F(m)z' \). Clearly, if \( |\sigma_k| = l \), then \( \bigcup_{i=1}^{l} \sigma_{ki} = \sigma_{k1} \).

Since \( g \) is an automorphism, we have

\[
(\forall x_i \in \sigma_{k1})(\forall y_j \in \sigma_{k1})(\exists r_i > 0)(g^{r_i}x_i = x_j \land\land g^{r_i}(F(m)x_i) = F(m)(g^{r_i}(x_i))).
\]
We will show that to any pair $\sigma_{k_i}$ and $\sigma_{k_j}$ from $\sigma_{k_1}$ we can associate one $r_{ij}$. Let $x_i, \bar{x}_i \in \sigma_{k_i}; x_j, \bar{x}_j \in \sigma_{k_j}$, and let $r$ and $R$ be such that $g^r x_i = x_j; g^R \bar{x}_i = \bar{x}_j$. Then, noting that $g^r$ and $g^R$ are automorphisms, we have

$$g^r(F(m x_i)) = F(m g^r(x_i)) = F(m x_j)$$

$$g^R(F(m \bar{x}_i)) = F(m g^R(\bar{x}_i)) = F(m \bar{x}_j).$$

But since $x_i, \bar{x}_i \in \sigma_{k_i}, x_j, \bar{x}_j \in \sigma_{k_j}$, we have $F(m x_i) = F(m \bar{x}_i); F(m x_j) = F(m \bar{x}_j)$, and thus $r \equiv R \pmod{\mu}$, where $\mu > 0$ is such that $g^{\mu+1} = g$.

Thus, for any pair of classes $\sigma_{k_i}$ and $\sigma_{k_j}$ there exists $r_{ij}$ such that $g^{r_{ij}}(\sigma_{k_i}) = \sigma_{k_j}$. If $g^{r_{ij}}$ is a bijection, then $|\sigma_{k_i}| = |\sigma_{k_j}|$, and thus there exists $p > 0$ such that $F(m)^p \sigma_{k_1} = p \sigma_{k_2}$.

We will show that all the copies $\sigma_{k_2}$ in $F(m)\sigma_{k_1}$ are ordered in the same way and that the order on them is identical with the initial order up to a cyclic shift.

The proof is by contradiction.

Let $|\sigma_{k_1}| = n; |\sigma_{k_2}| = l; n = pl$. Consider $x, \bar{x} \in \sigma_{k_1}$ such that $x \neq \bar{x}; F(m)x = F(m)\bar{x}; F(m)\bar{x} \in \sigma_{k_2}$. Here $\sigma_{k_2} = \sigma_{k_2}$, and the orders on these classes as subsets of $F(m)\sigma_{k_1}$ are different.

Let $F(m)x$ have the index $i_1$ in $\sigma_{k_2}$ and $F(m)\bar{x}$ the index $i_2$. Then $r = s l + i_2 - i_1$ is such that $g^r x = \bar{x}$. Since $g^r$ is an automorphism, we have

$$F(m)\bar{x} = F(m g^r(x)) = g^r(F(m x)) = g^{r_{ij}}(F(m x)) \neq F(m x),$$

because $|i_2 - i_1| < l$, which contradicts the choice of $x$ and $\bar{x}$.

Thus,

$$F(x) = g^{r_{ij}}(x) = g^{r_{ij}}(F(m x)) = g^{r_{ij}}(F(m x)) \neq F(m x).$$

Since $x$ is arbitrary, relationship (4), and thus also (2), are satisfied for all $m$. By arbitrary choice of the class $\sigma_{k_1}$ from the partition $\rho$, relationship (2) holds for all $\sigma_k$ such that $|\sigma_k| > 1$.

Let $|\sigma_k| = 1$ and $\sigma_k = \{x\}$. Since $g(x) = x$ and $g$ is an automorphism, we have

$$g(Fx) = F(g(x)) = Fx.$$

But for every $\sigma_{k_i} \in X/\rho$ we have $g(\sigma_{k_i}) = \sigma_{k_i}$, and therefore any nonempty trace of the class $\sigma_{k_i}$ on $F \bar{x}$ is identical with the entire class $\sigma_{k_i}$.

Thus, $F \sigma_{k_i} = \{\sigma_{k_i}\}$ for $|\sigma_k| = 1$. Q.E.D.

**Sufficiency.** Let the partition $\rho$ be such that (2) holds for all $m$ and any $\sigma_k$ from $\rho$.

We will show that the partition $\rho$ induces an automorphism $g$. Extend the definition of the orders on the classes $\sigma_k$ to cyclic orders. As the permutation $g$ take the collection of cycles $\{\sigma_k\}$. For each class $\sigma_k$ from the partition $\rho$ we obviously have $g(\sigma_k) = \sigma_k$. And since $(F \sigma_k)_m = \rho \sigma_{k_i}$ for every $m > 0$, we have

$$g((F \sigma_k)_m) = \rho(g(\sigma_{k_i}^s)), $$

where $s$ is a cyclic shift by a certain number of digits. Since $g$ and $s$ are bijections, we have $g(\sigma_{k_i}^s) = (g(\sigma_{k_i}))^s$.

Consider the elements $x, x' \in \sigma_k, y, y' \in \sigma_k$, such that $g(x) = x'; F(m)x = y; F(m)x' = y'$.

Assume that the permutation $g$ associates to the element $y \in \sigma_{k_i}$ the element $y' \in \sigma_{k_i}$. Then noting that $g$ and $s$ are permutations and using (2), we see that the same element $y'$ is associated to the element $y \in \sigma_{k_1}; \sigma_{k_1} \in F \sigma_{k_1}$. Now, let $x$ from $\sigma_k$ have the index $\alpha_1$ and $x'$ the index $\alpha_2$. Similarly, $y \in \sigma_{k_1}$ has the index $\beta_1$ and $y'$ the index $\beta_2$. If $|\sigma_k| = n$: $|\sigma_{k_1}| = l; n = pl$, then $r = s l + \beta_2 - \beta_1 = \alpha_2 - \alpha_1$ is the number of one-digit shifts $q$ needed to obtain $x'$ from $x$.

Clearly, $g(x) = g^q(x) = x'$. On one hand, $F(m)(g^q(x)) = F(m)(x') = y', g(y) = y'$, and on the other hand for $y$ from $\sigma_{k_1}$ by (2)

$$g(y) = g^r(y) = g^{\beta_2 - \beta_1}(y) = y'. $$

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