IMPLICIT SINGLE-SEQUENCE METHODS FOR INTEGRATING ORBITS

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(Received 24 May, 1973)

Abstract. The solutions of \( \ddot{x} = F(x, t) \), and also \( \ddot{x} = F(x, t) \), are developed in truncated series in time \( t \) whose coefficients are found empirically. The series ending in the \( t^6 \) term yields a position at a final prechosen time that is accurate through 9th order in the sequence size. This is achieved by using Gauss-Radau and Gauss-Lobatto spacings for the several substeps within each sequence. This time-series method is the same in principle as implicit Runge-Kutta methods, and the present algorithm generates coefficients for families of implicit Runge-Kutta forms, including some not described previously. In some orders these methods are unconditionally stable (A-stable). In the time-series formulation the implicit system converges rapidly. For integrating a test orbit the method is found to be about twice as fast as high-order explicit Runge-Kutta-Nyström-Fehlberg methods at the same accuracies. Both the Cowell and the Encke equations are solved for the test orbit, the latter being 35% faster. It is shown that the Encke equations are particularly well-adapted to treating close encounters when used with a single-sequence integrator (such as this one) provided that the reference orbit is re-initialized at the start of each sequence. This use of Encke equations is compared with the use of regularized Cowell equations.

1. Introduction

The method of integrating \( \ddot{x} = F(x, t) \) and \( \ddot{x} = F(x, t) \) described here was developed from time-series expansions as a practical and fast single-sequence integrator. Later this was found to be related to the implicit Runge-Kutta method, although the formulations and algorithms of the two methods are entirely different.

From the time-series approach we are concerned with methods wherein the series is fitted to the function \( F \) at several computed points, and in particular to those cases where these points are not evenly spaced. Such methods have been developed by Wielen (1967) and Aarseth (1972). They fit an empirical polynomial in time through the forces found at several previous steps which can be unevenly spaced. Integrating this fitted curve they predict the position at the next step, determine the force there, and then correct this position. Theirs is a multi-step method with variable step size.

Our method is similar in that we also integrate a time series found by fitting an empirical curve through forces at several unevenly-spaced points, but it is different in several respects: (1) It is self-starting. (2) All the points are at the current position or in advance of it. (3) There are several forward substeps taken during an integration sequence before the final corrected position is found. It is a single-sequence method.

In the above respects our method has a pattern like that of Runge-Kutta integrations. In fact, the most extensive comparisons to be presented are with the high-order explicit Runge-Kutta-Nyström methods developed by Fehlberg (1972).

An attractive aspect of our method is that one can obtain accuracies several orders higher than would be expected from the order of the fitting polynomial. This is achiev-
ed by adopting Gauss-Radau or Gauss-Lobatto spacings for the substeps. As in Butcher’s (1964) implicit Runge-Kutta method and Beaudet’s (1972) multi-off-grid method, we apply the principles of Gaussian quadratures to integrating differential equations.

Section 2 below derives the equations fundamental to the method, and Section 3 describes the integration procedure. Section 4 discusses Radau and Lobatto spacings, giving a detailed example of their applicability. There is also a discussion of why Gauss-Legendre spacings are not as useful here. Section 5 shows that the same procedures apply to both first- and second-order differential equations. An analysis shows that there is unconditional stability ($A$-stability) in the Lobatto cases when solving $\ddot{x} = F(x, t)$. Section 6 considers the correspondence with implicit Runge-Kutta methods, showing that the present algorithm develops the coefficients for four families of such methods. One of these families is $A$-stable and another is the same as one described by Butcher (1964). Section 7 describes numerical tests and comparisons with other methods of orbit integration. Finally, Section 7 points out that single-sequence integrators are particularly suitable for solving the Encke orbit equations because they allow rectification at the start of every sequence. The advantage shows in the case of orbits involving close encounters, and this is compared with the alternative of using regularized Cowell equations.

2. Fundamental Equations

The orbit equations of celestial mechanics are of the form

$$\ddot{x} = F(x, y, z, x_i, y_i, z_i, x_j, y_j, z_j, ..., t), \quad (1)$$

where the function $F$, which may be called the force, depends on time $t$ and the position $x, y, z$ of the body whose path is to be integrated, as well as the positions of other bodies, as identified by subscripts $i, j, ...$. There are 3 such equations for each body. It is sufficient to treat the solution of the class IIS equation ($S$ for special, since $\dot{x}$ is absent)

$$\ddot{x} = F(x, t), \quad (2)$$

since the extension to any number of simultaneous equations of the form of Equation (1) is a very well known procedure. The class I equation is treated in Section 5.

At the start of a sequence we reset time $t_1 = 0$, and the initial position $x_1$, velocity $\dot{x}_1$, and force $F_1$ are known. A time-series expansion of $F$ about time zero is

$$F = F_1 + A_1 t + A_2 t^2 + A_3 t^3 + \cdots + A_N t^N \quad (3)$$

Integrating Equation (3) one has

$$x = x_1 + \dot{x}_1 t + F_1 t^2/2 + A_1 t^3/6 + \cdots + A_N t^{N+1}/((N + 1)(N + 2)), \quad (4)$$

$$\dot{x} = \dot{x}_1 + F_1 t + A_1 t^2/2 + A_2 t^3/3 + \cdots + A_N t^{N+1}/(N + 1). \quad (5)$$

The truncated series of Equation (3) is not a Taylor series because the coefficients $A$