Nondifferentiable Mathematical Programming and Convex–Concave Functions

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Abstract. For a convex–concave function \( L(x, y) \), we define the function \( f(x) \) which is obtained by maximizing \( L \) with respect to \( y \) over a specified set. The minimization problem with objective function \( f \) is considered. We derive necessary conditions of optimality for this problem. Based upon these necessary conditions, we define its dual problem. Furthermore, a duality theorem and a converse duality theorem are obtained. It is made clear that these results are extensions of those derived in studies on a class of nondifferentiable mathematical programming problems.

Key Words. Minimax theorems, Kuhn–Tucker optimality conditions, duality theorem, converse duality theorem, constraint qualifications.

1. Introduction

In Ref. 1, Mond considered the following nondifferentiable programming problem:

\[
\min_x k(x) + (x' B x)^{1/2}, \\
\text{subject to } g(x) \leq 0, 
\]

where \( k \) and \( g \) are convex and differentiable functions from \( \mathbb{R}^n \) into \( \mathbb{R} \) and \( \mathbb{R}^p \), respectively, and \( B \) is an \( n \times n \) symmetric, positive-semidefinite matrix. According to Ref. 2, this problem can be reduced to

\[
\min_x \max_{y \in C} \{k(x) + x'y\}, 
\]

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subject to $g(x) \leq 0$, \hspace{1cm} (2-2)

where

$$C = \{Bw \mid w'Bw \leq 1\}.$$  

It is known that the set $C$ is a compact convex subset in $\mathbb{R}^n$ (see Ref. 3).

Under this aspect, Schechter (Ref. 2) treated the following slightly generalized problem:

$$\min_x k(x) + s(x|C), \text{ subject to } g(x) \leq 0. \hspace{1cm} (3)$$

Here, $C$ is any compact convex subset in $\mathbb{R}^n$ and $s(x|C)$ is the support function of $C$, i.e.,

$$s(x|C) = \max_{y \in C} x'y.$$  

Schechter defined its dual problem and established a duality theorem.

The purpose of this paper is to propose a more general problem than the preceding ones and to extend the results obtained in the studies of problems (1) and (3). The problem considered is as follows [Problem (P)]:

$$(P) \min_x \max_{y \in C} L(x, y),$$

subject to $g(x) \leq 0$;

here, $L$ is a convex-concave function, i.e., $L(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a convex function of $x$ for each $y$ and a concave function of $y$ for each $x$; $C$ is a compact convex subset in $\mathbb{R}^m$; and $g$ is as above. Then, taking

$$L(x, y) = k(x) + x'y,$$

problems (1) and (3) become special cases of Problem (P).

In Section 2, we obtain optimality conditions (Theorem 2.1) for Problem (P). It is seen that these conditions are intimately related to those of Ref. 4. Our theorem is proved by using a minimax theorem. See also Section 5.

In Sections 3 and 4, we define a dual problem to Problem (P) and establish a duality theorem (Theorem 3.1) and a converse duality theorem (Theorem 4.1). It should be noted that the convexity–concavity of $L$ is fairly effective in proving the converse duality theorem. These theorems are generalizations of the results of Mond’s and Schechter’s paper.

We shall denote by (CQ1) the set of the following constraint qualifications: (i) Kuhn–Tucker constraint qualification, (ii) Arrow–Hurwicz–Uzawa constraint qualification, and (iii) reverse convex constraint qualification; and we shall denote by (CQ2) the set of the following...