A Characterization of Chebyshev Curves in $\mathbb{R}^n$

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Abstract. Given a totally ordered set $T$ containing at least $n+1$ elements (say a subset of $\mathbb{R}^i$), the graph of the function $a: T \rightarrow \mathbb{R}^n$ is called a Chebyshev curve (in $\mathbb{R}^n$) if the determinant of the matrix $(a(t_1), a(t_2), \ldots, a(t_n))$ is either positive whenever $t_1 < t_2 < \cdots < t_n$ or negative whenever $t_1 < t_2 < \cdots < t_n$. For finite $T$ a characterization of these curves (sequences) has been given by the author.

In this paper the result is extended to non-finite $T$. The characterization proved here is an improved (reformulated) version of that given by the author for infinite $T$.


1. Introduction

For motivation and 'historical' background of the matter, see [1], [2]. For more recent developments, see Section 3.

Let $T$ be a totally ordered set, i.e. a set possessing an order '<' such that for any two different elements $t_1, t_2 \in T$ we have either $t_1 < t_2$ or $t_2 < t_1$. (We shall use '<' also for real numbers; the meaning will be clear from the context.)

$T$ is assumed to have at least $n+1$ elements, $n \geq 2$. The function $a: T \rightarrow \mathbb{R}^n$, more precisely its graph $\{a(t)\}_{t \in T}$, is called a Chebyshev (C)-curve in $\mathbb{R}^n$ if

$$\text{sgn}(\det(a(t_1), \ldots, a(t_n))) = \text{const} \neq 0$$

for all $t_1 < t_2 < \cdots < t_n$, where $\det(\cdot)$ means the determinant of the matrix and $\text{sgn}(\cdot)$ is the signum function ($\text{sgn}(\alpha) = 0$ if $\alpha = 0$, $= 1$ if $\alpha > 0$ and $= -1$ if $\alpha < 0$).

The typical classical example for a Chebyshev curve is the generalized moment curve:

$$a(t) := (t^\alpha, t^\beta, \ldots, t^\nu), t > 0, \alpha < \beta < \cdots < \nu.$$  \hfill (1.2)

To prove that (1.2) satisfies (1.1) one has only to observe that the underlying matrix is the generalized Van der Monde matrix ([3]).

Using functions $e^{-\alpha t^2}$ or $1/t$ or $1/\text{ch}(t)$ (instead of $t^\alpha$), one can build three more matrices behaving similarly, see, e.g., [3, pp. 88–95], so one get three further examples of C-curves.
In fact the curve (1.2) and also those with $e^{-\sigma t^2}$, $1/t$, $1/\text{ch}(t)$, satisfy a more strict condition: not only (1.1) is valid with $\text{const} = +1$ but also all subdeterminants of all orders are positive ([3, pp. 88–95]). Matrices of such type are called totally positive ([3], [4], [5]). Let us call the curves $a: T \to R^n$ such that the matrices $(a(t_1), \ldots, a(t_n))$ are either for all $t_1 < \cdots < t_n$ totally positive or for all $t_1 < \cdots < t_n$ totally negative, totally monotone (tm)-curves in $R^n$.

So any tm-curve is a C-curve, but in general the converse is not true.

The curve (1.2) with $\alpha = 1, \beta = 2, \ldots, \nu = n$, (the moment curve of Carathéodory) is not only the classical example for a tm-curve but it served as the ‘paradigm’ for a so-called n-order curve (see, e.g., [6]).

Following [6], we call the curve $a: T \to R^n$ an n-order curve in $R^n$ if

$$\text{sgn} \left( \det \left[ \begin{array}{c} 1 \\ a(t_1) \\ a(t_2) \\ \ldots \\ a(t_{n+1}) \end{array} \right] \right) = \text{const} \neq 0$$

(1.3)

for all $t_1 < t_2 < \cdots < t_{n+1}$.

While the relation between tm-curves and C-curves is clear, that between C-curves and n-order curves is not so clear (notice principal differences between (1.1) and (1.3)).

Looking at the function $a: T \to R^n$ ‘row-wise’, i.e. as (an ordered) collection of $n$ functions $a_i: T \to R^1$, $i = 1, 2, \ldots, n$, where $a_i$ are the coordinates of $a \in R^n$, C-curves are nothing else but Chebyshev systems.

For basic facts concerning these systems see [3], [4], [5], [7]. For us the following well-known statement concerning Chebyshev systems is interesting (formulated in our terminology of C-curves)

**Proposition 1.1** (e.g. [4]). If $a: T \to R^n$ is a C-curve, then, denoting by $\langle x, y \rangle$ the usual scalar product of $x, y \in R^n$ and putting

$$\langle a, x \rangle(t) := \langle a(t), x \rangle, \quad t \in T, \quad x \in R^n,$$

(1.4)

for any non-zero $x \in R^n$ we have

$$S^+ (\langle a, x \rangle) \leq n - 1,$$

(1.5)

where $S^+(f)$ is the maximum number of sign changes of the function $f: T \to R^1$ on $T$.

Recall that for any $f: T \to R^1$

$$S^+(f) := \sup S^+(f(t_1), f(t_2), \ldots, f(t_p)),$$

(1.6)

where $S^+(\alpha_1, \ldots, \alpha_p)$, $\alpha_i \in R^1$, means the maximum number of sign changes in the sequence of signs $\text{sgn}(\alpha_1), \text{sgn}(\alpha_2), \ldots, \text{sgn}(\alpha_p)$ and the supremum is taken