ABSTRACT. For convex bodies $D$ in $\mathbb{R}^n$ it is shown that the isoperimetric deficit of $D$ is minorized by a constant times the square of the barycentric asymmetry $\beta(D)$ of $D$. Here $\beta(D)$ is defined as the volume of $D \setminus B_D$ divided by the volume of $D$, where $B_D$ denotes the ball centred at the barycentre of $D$ and having the same volume as $D$.

Consider the isoperimetric deficit

$$\Delta(D) = \frac{S(D)}{S(B_D)} - 1$$

of a bounded measurable set $D \subset \mathbb{R}^n$ whose boundary $\partial D$ has a finite $(n-1)$-dimensional surface area $S(D)$. Here $B_D$ denotes a ball in $\mathbb{R}^n$ with the same volume as $D$: $V(B_D) = V(D)$.

In a previous paper [1] the isoperimetric deficit $\Delta(D)$ of convex domains $D$ in $\mathbb{R}^n$ was estimated from below in terms of the uniform spherical deviation $d(D)$ of $D$. Assuming, as we may, that the volume of $D$ equals that of the unit ball in $\mathbb{R}^n$, $d(D)$ is simply the Hausdorff distance between $D$ and the ball of radius 1 centred at the barycentre of $D$. The main result in [1] then asserts that there exist constants $k_n, \eta_n > 0$ depending only on $n$ such that

$$\Delta(D) \geq \begin{cases} k_2 d(D)^2, & n = 2 \\ k_3 d(D)^2 / \log(1/d(D)), & n = 3 \\ k_4 d(D)^{(n+1)/2}, & n \geq 4 \end{cases}$$

for any convex body $D$ in $\mathbb{R}^n$ such that $\Delta(D) < \eta_n$. This estimate is sharp with regard to the order of magnitude of the function of $d(D)$ on the right.

A qualitative consequence of (1) is the following stability property connected with the isoperimetric theorem: If the isoperimetric deficit $\Delta(D)$ of a convex body $D$ is small enough then the uniform spherical deviation $d(D)$ is necessarily as small as we please.

In the planar case $n=2$ the estimate (1) is a variant of Bonnesen's inequality. This inequality extends to non-convex, simply connected domains, see [2], but not to disconnected domains, where also the above

stability property breaks down (take for $D$ the union of two discs of very different radii and with centres far apart). And for $n \geq 3$ stability breaks down for non-convex domains $D$, even homeomorphic to a ball (take for $D$ a ball equipped with a long, thin 'spike').

In an effort to obtain a kind of stability even for non-convex sets, L. E. Fraenkel has proposed to replace the above uniform spherical deviation $d(D)$ by the following average measure of non-sphericity of $D$, called the asymmetry of $D$:

$$
\alpha(D) = \inf_{x \in \mathbb{R}^n} \frac{V(D \setminus B(x, v))}{V(D)} = \inf_{x \in \mathbb{R}^n} \frac{V(B(x, v) \setminus D)}{V(D)},
$$

where $v$ denotes the volume radius of $D$ (=the radius of a ball with the same volume as $V$), and $B(x, v)$ denotes the ball of radius $v$ centred at $x$. Fraenkel conjectured the inequality

$$
\Delta(D) \geq c_n \alpha(D)^2
$$

with $c_n > 0$ depending only on the dimension $n$.

In dimension $n = 2$ the inequality (3) holds with $c_2 = 0.08$, as shown in [4]. For $n \geq 3$ the problem is open in its general form, but it was shown by Hall [3] that we always have $\Delta(D) \geq c_n \alpha(D)^4$, and so there is indeed stability when the non-sphericity of $D$ is measured by the asymmetry $\alpha(D)$. It was further shown in [3] that (3) holds as it stands for bodies of revolution $D$ (suitably understood), and it was noted that the exponent 2 on the right of (3) is sharp (take for $D$ a solid ellipsoid of revolution close to a ball in $\mathbb{R}^n$). Here and in the sequel the value of $c_n$ may vary from occurrence to occurrence.

In the present note we establish (3) for convex bodies $D$ in $\mathbb{R}^n$, $n \geq 2$. Actually we even replace $\alpha(D)$ in (3) by a generally larger quantity $\beta(D)$ which we call the barycentric asymmetry of $D$ because it is defined by fixing the centre $x$ of the ball $B(x, v)$ in (2) as the barycentre of $D$, cf. (5) below. This sharpening, however, is confined to the convex case, see Remark 1 below. Our proof makes extensive use of the proof of (1) given in [1].

None of the estimates (1) or (3) for convex domains $D$ in $\mathbb{R}^n$ can be derived from the other by inserting mutual estimates of $d(D)$ and $\alpha(D)$ (or $\beta(D)$), although such estimates are available in a sharp form, cf. Lemma and Remark 2 below.

NOTATION. The notation will be as in [1, §1, pp. 622–623]. In particular:

- $D$ = a convex body in $\mathbb{R}^n$, $n \geq 2$.
- $V = V(D)$ = the volume of $D$ ($n$-dimensional Lebesgue measure).
- $S = S(D)$ = the surface area of $D$ (i.e. of $\partial D$).