ON ARCS IN PATH DESIGNS OF BLOCK SIZE 3

ABSTRACT. For each admissible v we exhibit a path design $P(v, 3, 1)$ with a spanning set of minimum cardinality and a $P(v, 3, 1)$ with a scattering set of maximum cardinality. Moreover, we study maximal independent sets (or complete arcs in the geometric terminology) having the minimum number of secants, i.e. sets which are both spanning and scattering, and complete arcs with the maximum number of secants.

1. INTRODUCTION AND PRELIMINARIES

Let $K_v$ be the complete undirected graph on v vertices, and let $P_3$ be the simple path with two edges. A $P_3$-design of $K_v$ is a pair $(V, B)$, where $V$ is the vertex set of $K_v$ and $B$ is an edge-disjoint decomposition of $K_v$ into copies of the path $P_3$. As usual we call blocks the elements of $B$ and block set the set $B$. Moreover, given a block $b \in B$, we identify $b$ with its vertex set. A $P_3$-design of $K_v$ is also called a path design of order v and block size 3, and is denoted by $P(v, 3, 1)$.

Let $L$ be a set of edges of the complete graph $K_v$. A partial path design of order v and block size 3 is the decomposition of $K_v - L$ into copies of $P_3$. The edge set $L$ is called the leave of the partial path design.

A handcuffed design $H(v, 3, 1)$ is a path design $(V, B)$ such that each vertex of $V$ belongs to exactly r blocks. Obviously not every path design is balanced [4].

The necessary condition $v(v-1) \equiv 0 \pmod{4}$, $v \geq 3$ is also sufficient for the existence of a $P(v, 3, 1)$ [7].

A handcuffed design $H(v, 3, 1)$ exists if and only if $v \equiv 1 \pmod{4}$ [5].

Let $(V, B)$ be a path design $P(v, 3, 1)$. A subset $X$ of $V$ is called independent if no block of $B$ is contained in $X$. An independent set is maximal if for all $x \in V \setminus X$, $X \cup \{x\}$ is not independent. Independent sets have been studied geometrically as arcs. Of particular interest are maximal independent sets, or (in the geometric terminology) complete arcs (see, for example, [2], [3]).

A block $b \in B$ is called secant or tangent or exterior to $X$ if $|b \cap X| = 2$ or 1 or 0 respectively.

The spanned set $\mathcal{S}(X)$ is the set of $y \in V \setminus X$ such that there is at least one block secant to $X$ and meeting $y$. A set $X \subseteq V$ is a spanning set if for every $v \in V \setminus X$, $v \in \mathcal{S}(X)$. A scattering set $X \subseteq V$ is an independent set for which

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every \( y \in V \setminus X \) has the property that \( y \) appears in a block with at most two pairs of elements in \( X \).

It is straightforward to verify that every complete arc is a spanning set; the converse need not hold.

Spanning and scattering sets in Steiner triple systems are introduced by Colbourn et al. [1]. In this paper the authors exhibit, for every admissible \( v \), an \( \text{STS}(v) \) with a spanning set of minimum cardinality, and an \( \text{STS}(v) \) with a scattering set of maximum cardinality. In the process, they establish the existence of Steiner triple systems with complete arcs of the minimum possible cardinality. The analogous problem for handcuffed designs \( H(v, 3, 1) \) is completely solved in [6]. Moreover, Quattrocchi [6] studies complete arcs in \( H(v, 3, 1) \) having either the minimum number of secants (i.e. which are both spanning and scattering sets), or the maximum number of secants.

In this paper we exhibit, for each \( v \equiv 0, 1 \pmod{4} \), a \( P(v, 3, 1) \) with a spanning set of minimum cardinality and a \( P(v, 3, 1) \) with a scattering set of maximum cardinality. We also determine the spectrum of cardinalities of complete arcs having the minimum number of secants (i.e. which are both spanning and scattering sets) and the spectrum of cardinalities of complete arcs having the maximum number of secants.

Now we construct some partial \( P(v, 3, 1) \) designs which will be useful in the following. Given a non-negative integer \( t_v \) such that

\[
t_v \leq \binom{v}{2} \quad \text{and} \quad \binom{v}{2} - t_v
\]

is even, we set

\[
b_v = \frac{1}{2} \left[ \binom{v}{2} - t_v \right].
\]

**Lemma 1.** The complete graph \( K_v \) on \( v \) vertices can be decomposed into a partial \( P(v, 3, 1) \) \((V, B)\) with \(|V| = v, |B| = b_v \) and a leave of \( t_v \) edges.

**Proof.** Let \( V = \{1, 2, \ldots, v\} \). If \( v \equiv 3 \pmod{4} \), let

\[
T_i = \{\{j, j + 2i, j + 1\} : j = 1, 2, \ldots, v\} \quad \text{for } i = 1, 2, \ldots, \frac{v-3}{4},
\]

and

\[
T = \left\{ \left\{ j, j + \frac{v-1}{2}, j + v - 1 \right\} : j = 1, 2, \ldots, \frac{v-1}{2} \right\}.
\]

Put

\[
B_1 = \bigcup_{i=1}^{(v-3)/4} T_i \quad \text{and} \quad B = B_1 \cup T.
\]