Dido's Problem in the Plane for Domains with Fixed Diameter*

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Abstract. We find the connected compact domains in the closed half-plane, with fixed area and diameter, which minimize the relative perimeter.


1. Introduction

As in the case of the isoperimetric problem, one can also impose constraints on Dido's problem, that is the isoperimetric problem relative to a subset of $\mathbb{R}^n$ (see [2] for definitions and [3] for a survey of different kinds of isoperimetric problems). In [5] we solved the Dido problem with fixed inradius. Now we treat the problem of fixed diameter.

We shall consider connected compact domains $\mathcal{D}$ in the half-plane $\mathcal{H} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$.

Let $\mathcal{H}^+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, $\partial \mathcal{H} = \mathcal{H} - \mathcal{H}^+$. We shall put $d(A, B)$ to denote the distance between the points $A$ and $B$, $\text{Diam}(\mathcal{D})$ to denote the diameter of $\mathcal{D}$, $\text{Area}(\mathcal{D})$ for the Lebesgue measure of $\mathcal{D}$, $\text{Perim}(\mathcal{D})$ for the Minkowski content of $\partial \mathcal{D}$, and finally $\text{RelPerim}(\mathcal{D})$ for the relative perimeter of $\mathcal{D}$, that is $\text{Perim}(\mathcal{D})$ minus the one-dimensional Lebesgue measure of $\partial \mathcal{D} \cap \partial \mathcal{H}$.

Let $a > 0$ and $s > 0$. We shall consider the connected compact domains $\mathcal{D}$ in $\mathcal{H}$ such that $\text{Diam}(\mathcal{D}) = 2a$, $\text{Area}(\mathcal{D}) = s$ and answer the question whether among them there is one (up to a translation) with minimum relative perimeter.

If $0 < s \leq \frac{1}{2} \pi a^2$ then the answer is, as could be expected, a circular segment. The surprise comes for the case $\frac{1}{2} \pi a^2 < s < \pi a^2$. Then the solution is given by a domain defined by five circular arcs. Thus, we first describe the solutions.

Let $a > 0$ and $0 < s \leq \frac{1}{2} \pi a^2$. We denote by $\text{Segm}(a, s)$ the circular segment of area $s$ lying in $\mathcal{H}$ whose chord is the interval $[-a, a]$ of the $X$ axis (Figure 1).

Let $a > 0$ and $\frac{1}{2} \pi a^2 \leq s \leq \pi a^2$. Then, we shall denote by $\text{Oog}(a, s)$ the optimal ogive of diameter $2a$ and area $s$, that is the domain depicted in Figure 2. In

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it, the line $AF$ lies in the $X$ axis, the arcs $AB$ and $EF$ have a common center $Q$ and radius $a$, the line $GQ$ lies in the $Y$ axis, the center of the arc $DE$ (resp. $BC$) is $A$ (resp. $F$) and its radius is $2a$, and, finally, the arc $CD$ has its center in $P$, the point where the three lines $AD$, $FC$ and $BE$ meet. If $a$ is fixed, the choice of the angle $\alpha$, $0 \leq \alpha \leq \pi/2$, determines the ogive. If $\alpha = 0$, then it is the half circle of radius $a$; if $\alpha = \pi/2$, it is the circle of radius $a$.

We shall prove the following result.

**THEOREM.** Let $a, s > 0$. Let $\mathcal{D} \subset \mathcal{H}$ be a connected compact domain such that $\text{Diam}(\mathcal{D}) = 2a$, $\text{Area}(\mathcal{D}) = s$.

1. If $0 < s \leq \frac{1}{2} \pi a^2$, then $\text{RelPerim}(\text{Segm}(a, s)) \leq \text{RelPerim}(\mathcal{D})$ and the equality sign implies $\mathcal{D} = \text{Segm}(a, s)$ up to a translation along the $X$ axis.
2. If $\frac{1}{2} \pi a^2 < s \leq \pi a^2$, then $\text{RelPerim}(\text{Oog}(a, s)) \leq \text{RelPerim}(\mathcal{D})$ and the equality sign implies $\mathcal{D} = \text{Oog}(a, s)$ up to a translation.

Of course, by Bieberbach's inequality (see [2, p. 93]), if $s > \pi a^2$ there is no compact domain of diameter $2a$ with area $s$.

We can start from a convex compact domain $\mathcal{D}$, for if $\mathcal{D}$ were not convex, we could obtain a convex compact domain that is better than $\mathcal{D}$, i.e. with same area, same diameter and no greater relative perimeter.

In fact, it is easily proved that the convex hull $h\mathcal{D}$ of $\mathcal{D}$ has the same diameter and no greater relative perimeter. Suppose that $f \equiv \text{Area}(h\mathcal{D}) - \text{Area}(\mathcal{D}) > 0$. Let $PQ$ be a diameter of $h\mathcal{D}$ and let $A_1$, $A_2$ be the intersections of $h\mathcal{D}$ with the open half-planes determined by the line containing $P$ and $Q$. At least one of them, say $A_1$, does not intersect $\partial \mathcal{H}$. If $\text{Area}(A_1) > f$, then we can 'chop' from $A_1$, by means of a line parallel to $PQ$, a chunk of area $f$. If $\text{Area}(A_1) < f$, we consider the new domain $\mathcal{D}' = \overline{A_2}$, then move it so that its diameter $PQ$ lie in the $X$ axis, and then chop it as before. The resulting domain will obviously be convex and better than $\mathcal{D}$.

In the following, if $\mathcal{D} \subset \mathcal{H}$ is convex, we shall call $\partial \mathcal{D} \cap \partial \mathcal{H}$ the base of $\mathcal{D}$.