FACTORORIZATION INTO \( k \)-DIMENSIONAL LINEAR BIJECTIONS

**Abstract.** Let \( V \) be a vector space, \( k \in \mathbb{N} \) with \( k \leq \dim V \) and \( S_k := \{ \phi \in \text{GL}(V) | \dim V(\phi - 1) = k \} \). Then \( S_k \) generates \( \text{GL}_f(V) := \{ \pi \in \text{GL}(V) | \dim V(\pi - 1) \text{ is finite-dimensional} \} \) (with the exception that \( \dim V = 2 = k \) and the field is GF2). We study the length problem in \( \text{GL}_f(V) \) with \( S_k \) as set of generators.

1. **Introduction**

Let \( V \) be a right vector space of arbitrary dimension \( n \) over a skew-field \( K \). Call \( \pi \) a simple mapping if \( \pi \in \text{GL}(V) \) and \( \dim B(\pi) = 1 \), where \( B(\pi) = V(\pi - 1) \). Then \( \pi \) induces on the projective space \( PV \) a perspective collineation with center \( V(\pi - 1) \) and axis kernel \( V(\pi - 1) \). Every perspective collineation of \( PV \) is induced by a simple mapping.

Let \( S \) be a set of generators for a group \( G \) such that \( a^{-1} \in S \) for every \( a \in S \). Then every \( \pi \in G \) is a product of elements of \( S \), and the minimal number of factors occurring in such a product is called the length of \( \pi \), notation \( l(\pi) \). If \( G = \text{GL}_f(V) := \{ \pi \in \text{GL}(V) | \dim B(\pi) \text{ is finite} \} \) and \( S \) is the set of simple mappings then \( S \) generates \( G \) and \( l(\pi) = \dim B(\pi) \) for every \( \pi \in G \). Furthermore, one can often obtain a representation \( \pi = \sigma_1 \cdots \sigma_{l(\pi)} \) where \( \det \sigma_i = \lambda_i \) for given values \( \lambda_i \) (cf. [1], [2]).

For \( k \in \mathbb{N} \) with \( k \leq \dim V \) let \( S_k := \{ \sigma \in \text{GL}(V) | \dim B(\sigma) = k \} \) denote the set of \( k \)-dimensional mappings. Observe that \( S_1 \) is the set of simple mappings. For \( \pi \in \text{GL}_f(V) \setminus \{ 1 \} \) let \( l(\pi) := \min \{ s \in \mathbb{N} | \pi = \varphi_1 \cdots \varphi_s, \varphi_i \in S_k \} \cup \{ \infty \} \) denote the \( k \)-length of \( \pi \). We will study the length problem with \( S_k \) as set of generators. For \( \pi \in \text{GL}_f(V) \setminus \{ 1 \} \) and \( m := \dim B(\pi) \) we prove that \( l(\pi) = 2 \) if \( m < k \) and \( l(\pi) = \min \{ r \in \mathbb{N} | m \leq rk \} \) if \( k \leq m \) (with the only exception that \( K = \text{GF2} \) and \( \dim V = 2 = k \)); cf. 4.4. Furthermore, if \( k \) does not divide \( m \) or if \( \pi \) (cf. 2.8) is not a homothety then we can prescribe the determinants of all but one of the factors in a product \( \pi = \varphi_1 \cdots \varphi_i \), where \( \varphi_i \in S_k \) and \( l := l(\pi) \); cf. supplement to 4.4.

In order to make the article self-contained we collect basic tools in Section 2. We include Section 3 on products of simple mappings though the results are known ([1] and [2]).
2. BASIC FACTS

For \( \pi \in \text{End}(V) \) let \( B(\pi) := V(\pi - 1) \) and \( F(\pi) := \ker(\pi - 1) \). Then \( B(\pi) \cong V/F(\pi) \), hence \( \dim B(\pi) = \text{codim} F(\pi) \). Let \( \text{End}_f(V) := \{ \pi \in \text{End}(V) \mid \dim B(\pi) \in \mathbb{N}_0 \} \) (where \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \)).

2.1. REMARK. Let \( \pi \in \text{End}(V) \) and \( U \subseteq V \). If \( B(\pi) \subset U \) or \( U \subset F(\pi) \) then \( U \pi \subset U \).

2.2. REMARK. Let \( \pi \in \text{GL}(V) \). Then \( B(\pi) = B(\pi^{-1}) \) and \( F(\pi) = F(\pi^{-1}) \).

2.3. LEMMA. Let \( \alpha, \beta \in \text{End}(V) \). Then \( B(\alpha \beta) + B(\beta) = B(\alpha) + B(\beta) \). In particular, \( B(\alpha \beta) \subset B(\alpha) + B(\beta) \). Let \( \alpha, \beta \in \text{End}_f(V) \). Then \( \dim B(\alpha \beta) = \dim B(\alpha) + \dim B(\beta) \) if and only if \( B(\alpha \beta) = B(\alpha) \oplus B(\beta) \).

2.4. LEMMA. Let \( \alpha, \beta \in \text{End}(V) \). If \( B(\alpha) \cap B(\beta) = \emptyset \) then \( F(\alpha \beta) = F(\alpha) \cap F(\beta) \). Furthermore,

\[
\dim F(\alpha \beta) \leq \dim (F(\alpha) \cap F(\beta)) + \dim (B(\alpha) \cap B(\beta)).
\]

2.5. LEMMA. (a) Let \( \alpha, \beta \in \text{GL}(V) \) such that \( V = F(\alpha) + F(\beta) \). Then \( B(\alpha \beta) = B(\alpha) + B(\beta) \).

(b) Let \( \alpha, \beta \in \text{GL}_f(V) \) and \( B(\alpha) \cap B(\beta) = \emptyset \). Then

\[
B(\alpha \beta) = B(\alpha) \oplus B(\beta) \iff F(\alpha) + F(\beta) = V.
\]

2.6. SIMPLE MAPPINGS. Call \( \sigma \) simple if \( \sigma \in \text{GL}(V) \) and \( \dim B(\sigma) = 1 \). Let \( r \in V \) and \( v \in V^* \) (dual space). Then \( \sigma := \sigma(r, v) : V \rightarrow V, v \mapsto v + r(vv) \) is linear; \( \sigma \) is simple if and only if \( r \neq 0 \) and \( v \neq 0 \) and \( rv \neq -1 \). Suppose that \( \sigma \) is simple. We have \( B(\sigma) = \langle r \rangle \) and \( F(\sigma) = \ker(v) \). Clearly, \( B(\sigma) \subset F(\sigma) \) if and only if \( rv = 0 \). Then \( \sigma \) is called a transvection. Every simple mapping in \( \text{GL}(V) \) has the form \( \sigma(r, v) \) with \( r \neq 0 \) and \( v \neq 0 \) and \( rv \neq -1 \).

If \( \sigma(r, v) \) and \( \sigma(s, \omega) \) are simple mappings then \( \sigma(r, v) = \sigma(s, \omega) \) if and only if \( s = rv \lambda \) and \( \omega = \lambda^{-1}v \) for some \( \lambda \in K^* \). This yields in particular: \( rv = \lambda(s\omega)\lambda^{-1} \). For \( \mu \in K^* \) let \( \zeta \mu \) denote the conjugacy class of \( \mu \) in \( K^* \). Due to the previous observation one can assign to every simple mapping \( \sigma = \sigma(r, v) \) the conjugacy class \( (1 + rv) \) which will be called the type of \( \sigma \). If \( \sigma \) is simple and \( \zeta \mu = \text{type}(\sigma) \) then \( \sigma = \sigma(s, \omega) \) for some \( s \in V \) and \( \omega \in V^* \) such that \( 1 + s\omega = \mu \). If \( K \) is commutative then the type of \( \sigma \) is \( \det(\sigma) \).

2.7. LEMMA. Given \( \alpha \in \text{GL}(V) \) and \( U \subseteq V \) such that \( B(\alpha) \subset U \) and \( B(\alpha) \neq U \). Let \( \lambda \in K^* \). If \( \alpha = 1 \) and \( \lambda = 1 \) assume that \( \dim V \geq 2 \). Then \( U = B(\alpha \beta) = B(\alpha) \oplus B(\beta) \) for some \( \beta \in \text{GL}(V) \). If \( K \) is commutative then we can additionally achieve that \( \det(\beta) = \lambda \).

Proof. We may assume that \( \dim U = \dim B(\alpha) + 1 \).

Case \( \lambda = 1 \). Choose some \( r \in U \setminus B(\alpha) \) with \( F(\alpha) \neq \langle r \rangle \) [this is possible since