ON BLASCHKE’S EXTENSION OF BONNESEN’S INEQUALITY

ABSTRACT. W. Blaschke established a Bonnesen-style inequality for the relative inradius and circumradius of a planar convex body K with respect to another. We sharpen this inequality by considering the radii of the minimal convex annulus of K.

1. INTRODUCTION

Let K and C be plane convex bodies, i.e. compact convex sets in the Euclidean plane $\mathbb{E}^2$ with nonempty interior. Assume that the origin 0 of $\mathbb{E}^2$ is an interior point of C. Then

$$
\varepsilon_i(K, C) = \sup\{\varepsilon \geq 0: \varepsilon C + x \subseteq K, x \in K\},
$$

$$
\mathcal{R}_u(K, C) = \inf\{\varepsilon > 0: \varepsilon C + x \supseteq K, x \in K\}
$$

are the relative inner and outer radius of K with respect to C. If A denotes the area, then the mixed area $W(K, C)$ is defined by

$$
A(K + C) = A(K) + 2W(K, C) + A(C).
$$

W. Blaschke [3, pp. 33–36] proved in 1936 the following inequality:

$$
A(K) - 2\varepsilon_i(K, C) + t^2A(C) \leq 0 \quad \text{for } \varepsilon_i(K, C) \leq t \leq \mathcal{R}_u(K, C).
$$

A proof of (1) may also be found in [7]. The special case $C = B^2(=\text{unit circle})$ had already been proved by T. Bonnesen [5] in 1929. In this paper we want to improve (1) for a smooth and strictly convex body C which is $\mathcal{C}$-symmetric (i.e. $C = -C$). For this we consider

$$
\varepsilon_x(K, C) = \sup\{\varepsilon \geq 0: \varepsilon C + x \subseteq K\}
$$

$$
\mathcal{R}_x(K, C) = \inf\{\varepsilon > 0: \varepsilon C + x \supseteq K\},
$$

where $x \in K$.

In [14] it was shown:

LEMMA 0. Let $K \subset \mathbb{E}^2$ be a convex body and let C be a smooth and strictly convex body. Then there is a unique point $x_0 \in K$ such that the function

$$
F_x(K, C) = \mathcal{R}_x(K, C) - \varepsilon_x(K, C)
$$

attains its minimum.
REMARK. Without the assumptions on \( C \) in Lemma 0, the point \( x_0 \) is not necessarily unique and Lemma 1 of Section 2 does not hold as the following example shows:

Let \( K = \text{conv}\{(0,0), (2,0), (0,2)\} \) and \( C = \{(x, y)\mid 0 \leq x \leq 2, 0 \leq y \leq 2\} \). Then it can easily be seen that for each \( x_0 = (t, t), \frac{1}{2} \leq t \leq 1 \) we have \( F_{x_0}(K, C) = 1 \). Thus, in the following, \( C \) will denote an arbitrary but fixed \( C \)-symmetric smooth and strictly convex body. Moreover, for general dimensions (without the uniqueness) the result has also been proved by Heil in [9].

NOTATIONS. We abbreviate: \( \mathcal{R}_{x_0}(K, C) = \mathcal{R}_0(K, C) \), \( \mathcal{R}_{x_0}(K, C) = \mathcal{R}_0(K, C) \). Moreover, the abbreviations \( \text{bd}, \text{int} \) and \( \text{conv} \) stand for boundary, interior and convex hull.

Our main result is now

THEOREM 1. Let \( K \subset \mathbb{E}^2 \) be a convex body. Then

\[
A(K) - 2tW(K, C) + t^2A(C) \leq 0 \quad \text{for } \mathcal{R}_0(K, C) \leq t \leq \mathcal{R}_0(K, C).
\]

COROLLARY.

\[
(W(K, C))^2 - A(K)A(C) \geq (A(C))^2 \frac{(\mathcal{R}_0(K, C) - \mathcal{R}_0(K, C))^2}{4}.
\]

Clearly \( \mathcal{R}_0(K, C) \leq \mathcal{R}_1(K, C) \) and \( \mathcal{R}_u(K, C) \leq \mathcal{R}_0(K, C) \). Moreover, we prove the following

THEOREM 2. Let \( K \subset \mathbb{E}^2 \) be a convex body. Then

\[
\mathcal{R}_1(K, C) < 2\mathcal{R}_0(K, C),
\]

\[
\mathcal{R}_0(K, C) < \left(\frac{3}{2}\right)\mathcal{R}_u(K, C)
\]

and both inequalities are tight.

REMARKS. 1. Inequality (3) follows from (2) with \( t = \mathcal{R}_0(K, C) \), \( t = \mathcal{R}_0(K, C) \) and elementary calculation.

2. For \( C = B^2 \), Vincze [19] showed that:

\[
\mathcal{R}_1(K, B^2) < 2\mathcal{R}_0(K, B^2) \quad \text{and} \quad \left(\frac{\sqrt{3}}{2}\right)\mathcal{R}_0(K, B^2) \leq \mathcal{R}_u(K, B^2)
\]

and both inequalities are tight.

3. The special case \( C = B^2 \) of (2) and (3), i.e. for circular annulus, was already found by Bonnesen [5, pp. 60 and 69] who also improved inequality (3) [5, p. 74].