A CATEGORICAL GLIMPSE AT THE RECONSTRUCTION OF GEOMETRIES

Dedicated to Karl H. Hofmann, on the occasion of his 60th birthday

Abstract. The author's reconstruction method ['Reconstruction of incidence geometries from groups of automorphisms', Arch. Math. 58 (1992) 621-624] is put in a categorical setting, and generalized to geometries with an arbitrary number of 'types'. The results amount to saying that the reconstruction process involves a pair of adjoint functors, and that the class of those geometries that are images under reconstruction forms a reflective subcategory.

The present paper deals with the problem whether a given geometry (i.e. roughly spoken, an incidence structure together with a group of automorphisms) is determined by the action of the group. After results on flag-transitive groups on incidence structures with two types of objects (see [7] for further historical notes), a first attempt by the author led to results that were quite satisfactory for the case of partial planes (see Proposition (2.7) below), and in particular for stable planes (see [7, 4]). In a discussion about this previous result, K.H. Hofmann urged a categorical point of view. This categorical treatment led to new insights: some of the previous restrictions on the action of the group may be dropped, and the reconstruction process applies to geometries with an arbitrary number of types. Moreover, the categorical point of view helps the groups to their full rights: in fact, this paper presents an equivalence (in a strictly categorical sense) of 'sufficiently homogeneous' geometries and systems of subgroups.

1. A CATEGORY OF INCIDENCE STRUCTURES

(1.1) NOTATION. By Set and Gp we denote the familiar categories of sets and mappings, and of groups and group homomorphisms, respectively. The fact that a morphism (in any category) is monic (epic) shall be stressed by the notation \[ \rightarrow \] (\[ \rightarrow \rightarrow \]). An isomorphism shall be denoted by \[ \Longrightarrow \], which is not to be confused with \[ \Rightarrow \] (logical implication). A natural number is defined as the set of its predecessors: \[ n := \{m \mid m < n\} \]. Thus 0 is the empty set, and 1 = \{0\}. For each mapping \[ f: X \rightarrow Y \] and each subset \[ U \subseteq Y \] we write \[ f^{-1}(U) := \{x \mid f(x) \in U\} \].

We shall consider categories that are not balanced, i.e. a morphism that is

both monic and epic need not be an isomorphism. Hence the following classes of morphisms are of interest:

(1.2) DEFINITION. Consider any category.
(a) A monomorphism \( m \) is called \( \textit{extreme} \), if for each factorization \( m = h \circ g \) the fact that \( g \) is epic implies that \( h \) is an isomorphism:

\[
\begin{array}{ccc}
A & \xrightarrow{m} & B \\
\downarrow{id_A} & & \downarrow{id_B} \\
A & \xrightarrow{g} & X
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
A & \xrightarrow{m} & B \\
\downarrow{id_A} & & \downarrow{id_B} \\
A & \xrightarrow{g} & X
\end{array}
\]

(b) Dually, an epimorphism \( e \) is called \( \textit{extreme} \), if for each factorization \( e = h \circ g \) the fact that \( h \) is monic implies that \( h \) is an isomorphism:

\[
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow{g} & & \downarrow{id_B} \\
X & \xrightarrow{h} & B
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow{g} & & \downarrow{id_B} \\
X & \xrightarrow{h} & B
\end{array}
\]

It is easy to see from the definition that an extreme monomorphism is an isomorphism if, and only if, it is an epimorphism; and dually. Note that \( \text{Set} \) and \( \text{Gp} \) are balanced categories, where, in particular, each monic and each epic is extreme. We shall denote extreme monics and epics by the symbols \( \circlearrowleft \) and \( \circlearrowright \), respectively.

(1.3) DEFINITION. An \( \textit{incidence structure} \ (I, A) \) consists of a family \( A : T \to \text{Set} : t \mapsto A_t \) and a subset \( I \subseteq \prod_{t \in T} A_t \). For incidence structures \((I, A)\) and \((J, B)\), a morphism \( f = (F, f_T) : (I, A) \to (J, B) \) consists of a mapping \( f_T : T \to U \) (where \( U \) is the domain of \( B \)), and a family \( F : t \mapsto F_t \) with domain \( T \) such that \( F_t : A_t \to B_{f_T(t)} \) is a mapping (for each \( t \in T \)). Finally we require that there exists some mapping \( F_I : I \to J \) such that

\[
\text{(A)} \quad
\begin{array}{ccc}
I & \xrightarrow{i} & \prod_{t \in T} A_t \\
\downarrow{F_I} & & \downarrow{F_{f_T}} \\
J & \xrightarrow{j} & \prod_{u \in U} B_u
\end{array}
\quad \xrightarrow{\text{pr}_t} \quad
\begin{array}{ccc}
A_t & \xrightarrow{pr_t} & A_t \\
\downarrow{F_{f_T}} & & \downarrow{F_{f_T}} \\
B_{f_T(t)}
\end{array}
\]

is commutative for each \( t \in T \), where \( i \) and \( j \) denote the inclusion maps, and \( \text{pr}_t \) the projection. Every mapping \( F_t \) that meets these requirements shall be called \( \textit{suited} \). These data define a category, which we shall denote by \( \text{Inc} \). The