AUTOMORPHISM GROUPS OF DIFFERENTIABLE DOUBLE LOOPS

Dedicated to Karl H. Hofmann, on the occasion of his 60th birthday

ABSTRACT. In this paper, we study local and global topological loops as well as topological double loops having a differentiable structure such that the loop operations are differentiable. The main result states that the group of differentiable automorphisms of a differentiable double loop is compact with respect to the compact-open topology.

The automorphism group \( \Gamma \) of a locally compact connected double loop \( \mathcal{D} \) is a locally compact topological group, where the group \( \Gamma \) will always be provided with the compact-open topology. For a proof of this result see [1]. If \( \mathcal{D} \) is even a Cartesian field, in particular if \( \mathcal{D} \) is one of the classical double loops \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) or \( \mathbb{O} \), then \( \Gamma \) is a compact Lie group. In general, it is an open problem whether the group \( \Gamma \) is compact or a Lie group. But the four classical double loops are not merely topological double loops; they also possess a differentiable structure such that the loop operations are differentiable. In a recent paper, J. Kozma has shown that the automorphism group of a \( C^2 \)-loop can be embedded into a linear group (see [7] or Theorem (2.1) below). The aim of this paper is to give an appropriate definition of a differentiable double loop and to prove that the automorphism group of such a double loop can be embedded as a closed subgroup into a linear group. Using this embedding theorem, we are able to verify the compactness of the automorphism group.

1. DEFINITIONS AND NOTATION

(1.1) DEFINITION. A quadruple \( \mathcal{L} = (L, U, 0, +) \) is called a local H-space if the following conditions are satisfied:

(1) \( L \) is a topological space.
(2) \( U \) is an open neighborhood of the element \( 0 \in L \).
(3) There exists an open neighborhood \( V \) of \( 0 \) in \( U \) such that the map \( +: V \times V \to U \) is continuous.
(4) \( x + 0 = 0 + x = x \) for every \( x \in V \).

The neighborhood \( V \) is called the support of \( \mathcal{L} \).

A local $H$-space $\mathcal{L} = (L, U, 0, +)$ with support $V$ is called a local loop iff the following statements hold:

(5a) For all $a, x, y \in V$ the equation $a + x = a + y \in U$ implies that $x = y$.
(5b) For all $a, x, y \in V$ the equation $x + a = y + a \in U$ implies that $x = y$.

Note that in the definition of a local loop we do not require that the local inverses of the operation $+$ are continuous.

(1.2) DEFINITION. A local $H$-space $\mathcal{L} = (L, U, 0, +)$ is called a smooth local $H$-space of dimension $n$ iff

(1) $U$ is an $n$-dimensional $C^2$-manifold,
(2) there is an open neighborhood $V \subseteq U$ of $0$ such that $+: V \times V \to U$ is a $C^2$-mapping.

The neighborhood $V$ is again called the support of $\mathcal{L}$.

A local loop $\mathcal{L} = (L, U, 0, +)$ is called a smooth local loop iff $\mathcal{L}$ is a smooth local $H$-space.

Note that if $\mathcal{L} = (L, U, 0, +)$ is a smooth local loop, then $\mathcal{L} = (L, U', 0, +)$ is also a smooth local loop for any neighborhood $U' \subseteq U$ of the element $0$.

The following result of J. P. Holmes and A. A. Sagle ([4, Th. 1.1]) shows that in the differentiable case the notion of a local $H$-space coincides with the notion of a local loop.

(1.3) THEOREM. A smooth local $H$-space is always a (smooth) local loop.

(1.4) DEFINITION. Let $\mathcal{L} = (L, U, 0, +)$ be a smooth local loop of dimension $n$ with support $V$. A $C^2$-diffeomorphism $h: U \to \mathbb{R}^n$ satisfying $h(0) = 0$ is called a smooth coordinate system of $\mathcal{L}$. A smooth coordinate system $h$ of $\mathcal{L}$ is called canonical iff there is a star-shaped neighborhood $S \subseteq h(V)$ of $0$ such that for every $x \in S$ the relation

$$h^{-1}(x) + h^{-1}(x) = h^{-1}(2x)$$

is satisfied.

The main tool for our investigations is a recent result by J. Kozma, which we state explicitly.

(1.5) THEOREM. Every smooth local loop has a smooth canonical coordinate system.

For a proof see [6, Th. 1].