ARRANGEMENTS OF HYPERPLANES WITH PROPERTY D

ABSTRACT. We denote by $M(\mathcal{A}_C)$ the complement of the complexification of a real arrangement $\mathcal{A}$ of hyperplanes. It is known that there is a certain technical property, called property D, on real arrangements of hyperplanes such that: if a real arrangement $\mathcal{A}$ of hyperplanes is simplicial then $\mathcal{A}$ has property D, and if $\mathcal{A}$ has property D then $M(\mathcal{A}_C)$ is a $K(\pi, 1)$ space. Our main goal is to prove that: if $\mathcal{A}$ has property D then $\mathcal{A}$ is simplicial. We also prove that a quasi-simplicial arrangement is always simplicial.

1. INTRODUCTION

Let $\mathbb{K}$ be a field and let $V$ be a vector space over $\mathbb{K}$. An arrangement of hyperplanes in $V$ is a finite set $\mathcal{A}$ of hyperplanes of $V$ through the origin. We say that $\mathcal{A}$ is a complex arrangement (resp. a real arrangement) if $\mathbb{K} = \mathbb{C}$ is the field of complex numbers (resp. if $\mathbb{K} = \mathbb{R}$ is the field of real numbers). The complement of a complex arrangement $\mathcal{A}$ is

$$M(\mathcal{A}) = V - \left( \bigcup_{H \in \mathcal{A}} H \right).$$

This space is an open connected submanifold of $V$. The complexification of a real arrangement $\mathcal{A}$ is the complex arrangement in $\mathbb{C} \otimes V = V_C$ defined by

$$\mathcal{A}_C = \{ \mathbb{C} \otimes H = H_C \mid H \in \mathcal{A} \}.$$

Let $\mathcal{A}$ be a real arrangement of hyperplanes. We say that $\mathcal{A}$ is essential if $\bigcap_{H \in \mathcal{A}} H = \{0\}$. A chamber of $\mathcal{A}$ is a connected component of $V - (\bigcup_{H \in \mathcal{A}} H)$. The arrangement $\mathcal{A}$ is called simplicial if $\mathcal{A}$ is essential and all the chambers of $\mathcal{A}$ are open simplicial cones.

In [De], Deligne proved that 'if $\mathcal{A}$ is a simplicial arrangement, then the complement of the complexification of $\mathcal{A}$, $M(\mathcal{A}_C)$, is a $K(\pi, 1)$ space'. All known proofs of Deligne's result use a certain technical property of simplicial arrangements, property D (see [Co], [De], [Pa] and [Sa]). This property will be defined in Section 2. It is known that:

(i) 'if $\mathcal{A}$ is a simplicial arrangement, then $\mathcal{A}$ has property D' (see [De, Prop. 1.19], [Pa, Th. 3.1] and [Sa, Th. 31]),

(ii) 'if a real arrangement $\mathcal{A}$ of hyperplanes has property D, then $M(\mathcal{A}_C)$ is a $K(\pi, 1)$ space' (see [Pa, Th. 3.6] and [Sa, Th. 33]).
Our goal in this paper is to prove that 'if a real arrangement \( \mathcal{A} \) of hyperplanes has property D, then \( \mathcal{A} \) is a simplicial arrangement' (Theorem 2.1). In particular, property D does not produce any example of non-simplicial arrangement \( \mathcal{A} \) of hyperplanes such that \( M(\mathcal{A}_C) \) is a \( K(\pi, 1) \) space. Our proof of Theorem 2.1 is inspired by the proof of [BEZ, Th. 3.1].

A convex polytope \( P \) is the dual of a 1-neighbourly polytope if any two faces (1-codimensional facets) of \( P \) always meet on a 2-codimensional facet. Such a polytope is not necessarily a simplex, for example, any cyclic polytope \( C(n, d) \) is 1-neighbourly and its dual is not a simplex in general (see [Br, Chap. II, §14 and 15]). A real arrangement \( \mathcal{A} \) of hyperplanes is called quasi-simplicial if \( \mathcal{A} \) is essential and all the chambers of \( \mathcal{A} \) are open cones over duals of 1-neighbourly polytopes. All simplicial arrangements are obviously quasi-simplicial. Deligne's proof of 'if \( \mathcal{A} \) is a simplicial arrangement, then \( \mathcal{A} \) has property D' can be easily generalized to quasi-simplicial arrangements; namely, if \( \mathcal{A} \) is a quasi-simplicial arrangement, then \( \mathcal{A} \) has property D. It follows, by our main result, that all quasi-simplicial arrangements are simplicial. We will give a short prove of this fact in this paper (Theorem 3.1).

Our work is organized as follows. In Section 2 we prove that 'if a real arrangement \( \mathcal{A} \) has property D, then \( \mathcal{A} \) is a simplicial arrangement'. In Section 3 we prove that 'if \( \mathcal{A} \) is a quasi-simplicial arrangement, then \( \mathcal{A} \) is a simplicial arrangement'.

2. Property D

Let \( \mathcal{A} \) be an arrangement of hyperplanes in a real vector space \( V \). The hyperplanes of \( \mathcal{A} \) subdivide \( V \) into facets. We denote by \( \mathcal{F}(\mathcal{A}) \) the set of all facets. The support \( |F| \) of a facet \( F \) is the vector space spanned by \( F \). Every facet is open in its support. We denote by \( \bar{F} \) the closure of \( F \) in \( V \). There is a partial order in \( \mathcal{F}(\mathcal{A}) \) defined by \( F \leq G \) if \( F \subseteq G \). A chamber is a 0-codimensional facet. A face is a 1-codimensional facet. Two chambers \( C \) and \( D \) are adjacent if they have a common face. In other words, a chamber is a connected component of \( V - (\bigcup_{H \in \mathcal{A}} H) \) and two chambers \( C \) and \( D \) are adjacent if and only if they are separated by only one hyperplane \( H \in \mathcal{A} \) which is the support of the common face. We will denote by \( \mathcal{C}(\mathcal{A}) \) the set of chambers of \( \mathcal{A} \). A wall of a chamber \( C \) is the support of a face \( F < C \). Note that a wall is a hyperplane included in \( \mathcal{A} \).

A gallery of \( \mathcal{A} \) is a sequence \( G = (C_0, C_1, \ldots, C_n) \) of chambers such that \( C_{i-1} \) and \( C_i \) are adjacent for \( i = 1, \ldots, n \) (here we assume \( C_{i-1} \neq C_i \)). We call \( C_0 \) the begin of \( G \), \( C_n \) its end, and \( n \) its length. For two galleries \( F = (C_0, C_1, \ldots, C_m) \) and \( G = (D_0, D_1, \ldots, D_m) \) with \( C_m = D_0 \), we write